AMERICAN

Journal of Mathematics

EDITED BY

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WITH THE COÖPERATION OF

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AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

Πραγμάτων ἔλεγχος οὐ βλεπομένων

VOLUME XL, NUMBER 2

BALTIMORE: THE JOHNS HOPKINS PRESS

LEMCKE & BUECHNER, *New York.*
G. E. STECHERT & CO., *New York.*
E. STEIGER & CO., *New York.*
ARTHUR F. BIRD, *London.*
WILLIAM WESLEY & SON, *London.*
A. HERMANN, *Paris.*
MAYER & MÜLLER, *Berlin.*

APRIL, 1918

Entered as Second-Class Matter at the Baltimore, Maryland, Postoffice.

On the Representation of Functions in Series of the Form $\Sigma c_n g(x+n)$.*

By R. D. Carmichael.

Introduction.

The most important functions defined by the Ω- and $\bar{\Omega}$-series, whose properties I have investigated in previous memoirs,† are doubtless those which have in a half-plane a Poincaré asymptotic representation ‡ in descending power series; and, in particular, those in which the series depend on a defining function $g(x)$ which has the asymptotic form §

$$g(x) \sim x^{\mu-x} e^{\alpha+\beta x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots\right), \tag{1}$$

valid in a sector V including the positive axis of reals in its interior, and which is analytic in V for sufficiently large values of $|x|$. We shall now use the symbol $S(x)$ to denote the series

$$S(x)=\sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}. \tag{2}$$

It will be observed that $S(x)$ belongs to the class of series denoted by $\bar{\Omega}(x)$ in the preceding papers, and also to the more special class denoted by $\bar{\omega}(x)$.

In the present paper I make a contribution towards solving the problem ‖ of representing given functions in the form of series $S(x)$ depending on a

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* Presented to the American Mathematical Society (at Chicago), April 7, 1917.

† *Transactions American Mathematical Society*, Vol. XVII (1916), pp. 207–232; American Journal of Mathematics, Vol. XXXIX (1917), pp. 385–403. These papers will be referred to by the numbers I and II, respectively.

‡ Compare II, especially §§ 4 and 5.

§ It is convenient to introduce here a slight change in notation; it is one, however, which can cause no confusion.

‖ That this includes the fundamental problem of representing given functions in the form of series dependent on functions defined by linear homogeneous difference equations may be seen from the following considerations. A fundamental property of the leading functions $G(x)$ defined by such equations is that expressed in the asymptotic relation

$$G(x) \sim x^{ix} e^{\beta x}\left(1+\frac{b_1}{x}+\frac{b_2}{x^2}+\ldots\right)$$

valid in a right half-plane, i being an integer and $\beta, b_1, b_2, \ldots$ being constants. (Compare Birkhoff, *Transactions American Mathematical Society*, Vol. XII (1911), pp. 243–284; and Carmichael, *ibid.*, pp.

given defining function $g(x)$. In order to render this problem amenable to simple methods it is necessary to place further restrictions on $g(x)$. We have seen (II, § 5) that

$$\frac{g(x+n)}{g(x)} \sim x^{-n}\left(\beta_{0n}+\frac{\beta_{1n}}{x}+\frac{\beta_{2n}}{x^2}+\ldots\right), \quad \beta_{0n}=e^{(\beta-1)n}, \tag{3}$$

where $\beta_{0n}, \beta_{1n}, \beta_{2n}, \ldots$ are a set of numbers independent of x. From this it follows that we have an asymptotic relation of the form

$$\frac{g(x)}{g(x+n)} \sim x^{n}\left(\gamma_{0n}+\frac{\gamma_{1n}}{x}+\frac{\gamma_{2n}}{x^2}+\ldots\right). \tag{4}$$

Some of the restrictions mentioned are stated most conveniently in terms of the coefficients γ. They appear in Section 4. Others appear in Section 5.

In § 1 I derive, in simple form, a necessary and sufficient condition on the coefficients c_n implying the convergence of $S(x)$ in an appropriately determined right half-plane. In § 2 some remarks are made on the order of increase of the coefficients in certain Poincaré asymptotic series. In § 3 the fundamental relations between these coefficients and the coefficients c_n of the associated series $S(x)$ are determined. In § 4 I construct, in a particular manner, functions having given Poincaré asymptotic representations of a certain type, and incidentally point out some fundamentally important instances of the series $S(x)$ which have occurred in the recent literature. Finally, in § 5, I show how to transform the Borel integral sum of a divergent series into a series $S(x)$ and also into a certain natural generalization of such series, and indicate some wide ranges of applicability of these results.

§ 1. *Order of Increase of Coefficients c_n in a Converging Series $S(x)$.*

Let $S(x)$ be a series of the form (2) which converges at every non-exceptional point in some half-plane; that is, one whose convergence number is not $-\infty$. Then, from the corollary to Theorem XII, in Memoir I, it follows that a real constant r_1 (finite or equal to $-\infty$) exists such that

$$\limsup_{\xi=\infty} \frac{\log \Sigma_\xi |c_\nu g(\nu)|}{\xi} = r_1, \tag{5}$$

99–134, and AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1916), pp. 163–182). In case $i=-1$ the function $G(x)$ itself is of the same type as the function $g(x)$ in the text. In general a suitable value of $g(x)$ may be defined in terms of $G(x)$ by the relation

$$g(x)=\frac{G(x)}{[\Gamma(x)]^{i+1}},$$

or by any one of several similar expressions which readily come to mind. Thus, for every function of the type $G(x)$ there exist corresponding functions of the type $g(x)$ suitable for use in the formation of series $S(x)$.

where $\Sigma_\xi \beta_\nu$ stands for the sum of all β_ν whose suffix ν satisfies the relation

$$e^{E(\xi)} \leqq \nu < e^\xi, \tag{6}$$

$E(\xi)$ denoting the greatest integer not greater than ξ. From the last inequality we see that

$$\xi < E(\xi) + 1 \leqq 1 + \log \nu$$

for every positive value of ξ and corresponding value of ν. Hence, from (5), it follows that a constant r_2 exists such that

$$\log |c_\nu g(\nu)| < r_2(1 + \log \nu), \qquad \nu > 0.$$

Thence it follows readily that a constant r exists such that

$$|c_\nu g(\nu)| < \nu^r, \qquad \nu \geqq 2. \tag{7}$$

Relation (7) states a condition which is necessary if $S(x)$ is to converge at every non-exceptional point in some half-plane.

It may also be shown that (7) states a condition sufficient to ensure that $S(x)$ thus converges in some half-plane. For from (7) it is clear that

$$\Sigma_\xi |c_\nu g(\nu)| < \nu^\rho e^\xi,$$

where ν and ξ are related as in (6), and ρ is a positive number not less than r. From this one sees readily that the superior limit in the first member of (5) has a value different from $+\infty$. This, in connection with the corollary to Theorem XII, in Memoir I, at once yields the conclusion that $S(x)$ converges at every non-exceptional point in some half-plane.

Thus we are led to the following theorem:*

Theorem I. *A necessary and sufficient condition that the series $S(x)$ in* (2) *shall converge for every non-exceptional value of x in some right half-plane is that a constant r exists such that*

$$|c_\nu g(\nu)| < \nu^r, \qquad \nu \geqq 2.$$

In view of the asymptotic form of $g(\nu)$ this theorem is seen to be equivalent to the following corollary:

Corollary: *A necessary and sufficient condition that the series $S(x)$ in* (2) *shall converge for every non-exceptional value of x in some right half-plane is that a constant r exists such that*

$$|c_\nu| < \nu^\nu e^{-\nu R(\beta)} \nu^r, \qquad \nu \geqq 2.$$

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* By exactly the same argumentation one is led to precisely the same necessary and sufficient condition for the similar convergence of the series $\Omega(x)$ and $\bar{\Omega}(x)$ in the case when $k=1$ and $m=0$ or 1. The corollary to Theorem XII, in Memoir I, yields immediately the corresponding result for the case when k and m do not satisfy the conditions just stated.

It is easy to see that an equivalent statement is obtained if one replaces the inequality in this corollary by the following:

$$|c_\nu| < \nu!\, e^{\nu[1-R(\beta)]}\nu^r, \qquad \nu \geqq 2.$$

To prove this, one has only to employ the well-known fact that

$$(\nu+1)!\,\nu^{-\nu}e^{\nu}\nu^{\frac{1}{2}}$$

approaches a finite limit different from zero as ν approaches infinity.

§ 2. *Order of Increase of the Coefficients in the Poincaré Asymptotic Representation of $S(x)$.*

We have seen (II, Theorem II and § 5) that a function $S(x)$, defined by the series in (2), has a Poincaré asymptotic representation of the form

$$S(x) \sim \beta_0 + \frac{\beta_1}{x} + \frac{\beta_2}{x^2} + \ldots . \tag{8}$$

valid in that part of V which lies in the half-plane $R(x) \geqq -\lambda_1$, where λ_1 is less than the convergence number λ of $S(x)$, and that the coefficients β_ν are given by the formulae

$$\beta_\nu = c_0\beta_{\nu 0} + c_1\beta_{\nu-1,1} + c_2\beta_{\nu-2,2} + \ldots + c_\nu\beta_{0\nu}, \qquad \nu = 0, 1, 2, \ldots . \tag{9}$$

The order of increase of $|\beta_\nu|$ with respect to ν evidently depends upon the β_{ij} quite as much as upon the c_i. In case the β_{ij} satisfy the restrictive condition

$$|\beta_{\nu-k,k}| < Me^{kR(\beta)}\nu^{\nu+\sigma}k^{-k}, \tag{10}$$

it is easy to see (by aid of the corollary to Theorem I) that we have

$$|\beta_\nu| < \nu^\nu \nu^s, \qquad \nu \geqq 2, \tag{11}$$

s being an appropriately chosen quantity independent of ν.

For the special case in which $g(x+1)/g(x)$ is analytic at $x=\infty$, other inequalities similar to (10) and (11) may be obtained. We write

$$\frac{g(x+1)}{g(x)} \equiv \rho(x) = \frac{\rho_1}{x} + \frac{\rho_2}{x^2} + \frac{\rho_3}{x^3} + \ldots .$$

Since this series converges for some x, a positive quantity ρ exists such that

$$|\rho_r| < \rho^r.$$

We take ρ to be greater than unity. We may write,

$$\frac{g(x+n)}{g(x)} \equiv \rho(x)\rho(x+1)\ldots\rho(x+n-1) = \prod_{i=0}^{n-1}\left[\frac{\rho_1}{x+i} + \frac{\rho_2}{(x+i)^2} + \ldots\right]$$

$$= \prod_{i=0}^{n-1}\left[\frac{\rho_1}{x} + \frac{\rho_2-\rho_1 i}{x^2} + \frac{\rho_3 - 2\rho_2 i + \rho_1 i^2}{x^3} + \ldots\right].$$

If this last product is expanded in powers of $1/x$, it is easy to see that the coefficients in the resulting series are less in absolute value than the coefficients in the similar expansion of the product

$$\prod_{i=0}^{n-1}\left[\frac{\rho}{x}+\frac{\rho(\rho+i)}{x^2}+\frac{\rho(\rho+i)^2}{x^3}+\frac{\rho(\rho+i)^3}{x^4}+\ldots\right].$$

These coefficients are not decreased if in this product $\rho(\rho+i)^{m-1}$ is replaced by $(\rho+n)^m$. Thence, by the multinomial theorem, it follows readily that

$$|\beta_{\nu-n,n}|<(n+\rho)^\nu\Sigma\frac{(i_1+i_2+\ldots+i_n)!}{i_1!\,i_2!\ldots i_n!},$$

where the summation is taken subject to the condition

$$i_1+2i_2+3i_3+\ldots+ni_n=\nu;$$

whence it follows that

$$|\beta_{\nu-n,n}|<\nu^n(n+\rho)^\nu. \tag{12}$$

Thence, through (9), and the corollary to Theorem I, we see that

$$|\beta_\nu|<\nu^{3\nu}e^{-\nu R(\beta)}\nu^t, \qquad \nu\geqq 2, \tag{13}$$

t being an appropriately chosen real quantity.

§3. *Evaluation of the Constants* c_ν *in Terms of the Constants* β_ν.

From (9) it follows that the constants c_ν may be expressed directly in terms of the constants β_ν. It is more convenient for our purposes, however, to proceed as follows: In equation (4) replace n by $\nu-n$, where $n<\nu$, and then replace x by $x+n$. Thus we have

$$\frac{g(x+n)}{g(x+\nu)}\sim(x+n)^{\nu-n}\left(\gamma_{0,\nu-n}+\frac{\gamma_{1,\nu-n}}{x+n}+\frac{\gamma_{2,\nu-n}}{(x+n)^2}+\ldots\right). \tag{14}$$

Then from

$$\frac{g(x)}{g(x+\nu)}\left(\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\ldots\right)\sim\sum_{n=0}^{\infty}c_n\frac{g(x+n)}{g(x+\nu)},$$

by means of (4) and (14), we have

$$\gamma_{0\nu}\beta_{\nu+1}+\gamma_{1\nu}\beta_\nu+\gamma_{2\nu}\beta_{\nu-1}+\ldots+\gamma_{\nu+1,\nu}\beta_0=\frac{1}{\gamma_{01}}c_{\nu+1}+\gamma_{21}c_{\nu-1}$$
$$+\gamma_{32}c_{\nu-2}+\ldots+\gamma_{\nu+1,\nu}c_0,$$

on equating the coefficients of $1/x$ in the expanded asymptotic forms of the two members. Now it is easy to see that

$$c_0=\beta_0,\quad c_1=\beta_1. \tag{15}$$

Hence, if we write

$$\eta_\nu=\gamma_{0\nu}\beta_{\nu+1}+\gamma_{1\nu}\beta_\nu+\gamma_{2\nu}\beta_{\nu-1}+\dots\dots+\gamma_{\nu\nu}\beta_1,\ \nu>0; \qquad \eta_0=\beta_1, \tag{16}$$

we have

$$\frac{1}{\gamma_{01}}c_{\nu+1}+\gamma_{21}c_{\nu-1}+\gamma_{32}c_{\nu-2}+\dots\dots+\gamma_{\nu,\,\nu-1}c_1=\eta_\nu, \qquad \nu=1,2,\dots\dots$$

Combining this system with the second equation in (15), and solving for $c_{\nu+1}$ we have

$$c_{\nu+1}=\gamma_{01}\eta_\nu+\gamma_{01}^3\eta_{\nu-2}\begin{vmatrix} 0 & \gamma_{21} \\ \frac{1}{\gamma_{01}} & 0 \end{vmatrix}-\gamma_{01}^4\eta_{\nu-3}\begin{vmatrix} 0 & \gamma_{21} & \gamma_{32} \\ \frac{1}{\gamma_{01}} & 0 & \gamma_{21} \\ 0 & \frac{1}{\gamma_{01}} & 0 \end{vmatrix}$$

$$+\gamma_{01}^5\eta_{\nu-4}\begin{vmatrix} 0 & \gamma_{21} & \gamma_{32} & \gamma_{43} \\ \frac{1}{\gamma_{01}} & 0 & \gamma_{21} & \gamma_{32} \\ 0 & \frac{1}{\gamma_{01}} & 0 & \gamma_{21} \\ 0 & 0 & \frac{1}{\gamma_{01}} & 0 \end{vmatrix}-\dots\dots, \qquad \nu=1,2,3,\dots\dots, \tag{17}$$

the expansion ending with the term containing η_0. Here γ_{01} has the value

$$\gamma_{01}=e^{1-\beta}.$$

We shall use the symbol Δ_k for the determinant of order k in the second member of equation (17).

§ 4. *Construction of Functions Having Given Poincaré Asymptotic Representations of a Certain Type.*

In this section we show how to construct a function $f(x)$ having a given Poincaré asymptotic representation

$$f(x)\sim\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\dots\dots, \tag{18}$$

of a certain type. We confine our attention to the case in which the function $g(x)$ and the constants $\beta_0, \beta_1, \beta_2, \dots\dots$ are jointly subject to the condition that positive quantities M and σ, both independent of ν, exist such that

$$|\eta_\nu|<M\nu^\nu e^{-\nu R(\beta)}\nu^\sigma, \qquad \nu=1,2,3,\dots\dots, \tag{19}$$

η_ν having the definition given in (16). Moreover, we suppose that $g(x)$ is such that a positive quantity M_1, and a real non-negative quantity σ_1, both independent of k, exist such that

$$|\Delta_k|<M_1k^ke^{-k}k^{\sigma_1}, \tag{20}$$

where Δ_k denotes the determinant defined at the end of Section 3. Our central theorem here is the following:

THEOREM II. *Let* $\beta_0, \beta_1, \beta_2, \ldots,$ *be a given set of constants, and let* $g(x)$ *be an associated function such that relations* (19) *and* (20) *are satisfied. Then, if constants* $c_0, c_1, c_2, \ldots.$ *are determined in terms of* $\beta_0, \beta_1, \beta_2, \ldots,$ *by means of equations* (15) *and* (17), *the series*

$$\sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)} \tag{21}$$

converges absolutely at all non-exceptional points in the half-plane $R(x) > \tau+1$, *where* τ *is the greater of the two quantities* 0 *and* $\sigma+\sigma_1+R(\mu)$; *and the function* $f(x)$ *represented by this series satisfies the Poincaré asymptotic relation* (18), *the latter being valid in the greatest region in the x-plane which is common to V and the half-plane* $R(x) \geqq \tau+1+\varepsilon$, ε *being any positive constant.*

We prove first that part of the conclusion which relates to the convergence of the series in (21). From relations (17), (19), (20) we see readily that

$$|c_{\nu+1}| < M_2 \nu^\nu e^{-\nu R(\beta)} \nu^{\sigma+\sigma_1+1}, \tag{22}$$

where M_2 is a quantity independent of ν. From the corollary to Theorem I it follows now that the series in (21) is convergent in an appropriately chosen right half-plane. Moreover, the last inequality affords partial information as to the position of the line of absolute convergence. This may be shown as follows: From the corollary to Theorem XII, in Memoir I, we see that the abscissa μ of absolute convergence of the series in (21) is given by the relation

$$\mu = -\limsup_{\xi=\infty} \frac{\log \Sigma_\xi |c_\nu g(\nu)|}{\xi},$$

where $\Sigma_\xi \zeta_\nu$ stands for the sum of all ζ_ν whose suffix ν satisfies the relation

$$e^{E(\xi)} \leqq \nu < e^\xi.$$

From (22) and the asymptotic form of $g(\nu)$ it is easy to see that

$$|c_\nu g(\nu)| < M_3 \nu^{\sigma+\sigma_1+R(\mu)}, \qquad \nu = 2, 3, \ldots,$$

M_3 being an appropriately chosen constant. Hence we have

$$\Sigma_\xi |c_\nu g(\nu)| < M_3 \sum_{\nu=1}^{\eta_\xi} \nu^{\sigma+\sigma_1+R(\mu)} \leqq M_3 \eta_\xi^{\tau+1},$$

where η_ξ is the greatest value of ν, such that $\log \nu < \xi$ and τ is the greater of the quantities 0 and $\sigma+\sigma_1+R(\mu)$. Hence,

$$\mu \geqq -\limsup_{\xi=\infty} \frac{(\tau+1)\log \eta_\xi + \log M_3}{\xi} = -(\tau+1).$$

This completes the proof of the part of the theorem which refers to the convergence of the series in (21).

The portion of the conclusion of the foregoing theorem which relates to the asymptotic character of $f(x)$ is an immediate consequence of Theorem II, of Memoir II, and the convergence properties just established.

The foregoing theorem states conditions to ensure the convergence of the series in (21). A central part of these conditions is contained in relations (19). If $g(x)$ satisfies conditions associated with (1), and if, moreover, $g(x)/g(x+\nu)$ is a polynomial when ν is sufficiently large, then from (17), and the corollary to Theorem I, it is easy to see that *relation* (19) *affords a necessary and sufficient condition for the convergence of* (21); and hence for the construction, by means of a series (21), of a function $f(x)$ having the given Poincaré asymptotic representation (18). It is not difficult to determine other classes of functions $g(x)$ for which (19) plays a like rôle in respect to necessary and sufficient conditions.

An important special use of the result contained in Theorem II should be pointed out. In many investigations, particularly in the theory of differential equations and of difference equations, one is led to divergent power series when one seeks to obtain a suitable representation of a function which is to be determined. If for an appropriately chosen function $g(x)$ the coefficients $\beta_0, \beta_2, \beta_3, \ldots.,$ in the diverging power series thus arising satisfy the conditions imposed by relations (19), then it is clear that a suitable modification of the computations will enable one to obtain a convergent expansion (21) in place of the diverging power series. It may be anticipated with considerable confidence that this converging expansion will serve to define such a function as one is seeking. It is my intention later to present such applications of Theorem II as are here indicated. For the special class of factorial series, important applications of this character have already been made in the theory both of differential equations* and of difference equations,† and in more general aspects of the theory of functions.‡

We shall next point out some simple conditions which are sufficient to ensure that relations (19) and (20) shall be satisfied. In the treatment of these conditions we shall have need of Hadamard's fundamental theorem§ concerning an upper bound to the absolute value of a determinant. This theorem may be stated as follows:

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* Horn, *Mathematische Annalen*, Vol. LXXI (1912), pp. 510–532.

† Nörlund, *Rendiconti del Circolo Matematico di Palermo*, Vol. XXXV (1913), pp. 177–216.

‡ Watson, *Rendiconti del Circolo Matematico di Palermo*, Vol. XXXIV (1912), pp. 41–48.

§ Hadamard, *Bulletin des Sciences Mathématiques* (Darboux), Vol. XVII (1893), pp. 240–246.

If Δ is a determinant of order n in which a_{ij} is the element in the *i-th* row and the *j-th* column, then

$$|\Delta| \leqq \sqrt{r_1 r_2 \ldots . r_n}, \quad |\Delta| \leqq \sqrt{\rho_1 \rho_2 \ldots . \rho_n},$$

where

$$r_i = \sum_{j=1}^{n} |a_{ij}|^2, \quad \rho_i = \sum_{j=1}^{n} |a_{ji}|^2.$$

With this theorem in hand we may readily determine conditions implying relation (20). Thus, if $\gamma_{k+1,k}$ satisfies the condition

$$|\gamma_{k+1,k}| \leqq k^{\frac{1}{2}}, \qquad k = K, K+1, K+2, \ldots ., \tag{23}$$

where K is a fixed integer, it may easily be shown that (20) is satisfied. For, from the first inequality in Hadamard's theorem above, we see that

$$|\Delta_k| < \bar{M} \cdot k! = \bar{M} \cdot \Gamma(k+1),$$

$\bar{M}$ being an appropriately chosen quantity independent of k. By means of the well-known asymptotic formula for $\Gamma(x)$, namely,

$$\Gamma(x) \sim x^x e^{-x} x^{-\frac{1}{2}} \sqrt{2\pi} \left(1 + \frac{1}{12x} + \ldots . \right),$$

it is now easy to see that (20) is satisfied provided that M_1 and σ_1 are given appropriate values. Hence, relation (23) expresses a condition sufficient *to ensure that a relation of the form* (20) *is satisfied.*

In order to determine simple conditions ensuring that (19) is satisfied we observe that the asymptotic relation (1) implies that the quantity

$$\frac{g(\rho+\nu)}{g(\rho)} \nu^{\nu} e^{-\beta\nu} \nu^{\rho-}$$

approaches a finite value different from zero as n approaches infinity, provided that $x = \rho$ is a point at which $g(x)$ is analytic and different from zero. From this it follows at once that a relation of the form (19) is satisfied, provided that a positive quantity M' and a non-negative real quantity σ', both independent of ν, exist such that

$$|\eta_\nu| < M' \left| \frac{g(\rho)}{g(\rho+\nu)} \right| \nu^{\sigma'}. \tag{24}$$

Let us consider now the special case in which the series in (18) converges when $|x|$ is sufficiently large. Then through (16) we see that a positive number ρ exists such that

$$\frac{1}{\rho} |\eta_\nu| < |\gamma_{0\nu}| \rho^{\nu} + |\gamma_{1\nu}| \rho^{\nu-1} + |\gamma_{2\nu}| \rho^{\nu-2} + \ldots . + |\gamma_{\nu-1,\nu}| \rho + |\gamma_{\nu\nu}|,$$
$$\nu = 1, 2, 3, \ldots . .$$

Comparing this with (4) we see that condition (24) is obviously satisfied in the special case when the quantities $\gamma_{0n}, \gamma_{1n}, \gamma_{2n}, \ldots$ are all positive or zero for every positive integral value of n greater than some given value N. Other simple conditions under which (24) is satisfied will readily occur to the reader.

Again, it may be seen that a relation of the form (19) is satisfied in case a positive quantity M and a non-negative real quantity σ exist such that

$$|\gamma_{k\nu}\beta_{\nu-k+1}| < M\nu^\nu e^{-\nu R(\beta)}\nu^{\sigma-1} \tag{25}$$

for every ν and corresponding k not greater than ν. For determining whether (25) is satisfied one may make use of the relation

$$\gamma_{k\nu} = \frac{1}{2\pi i}\int_C \frac{f_\nu(x)\,dx}{x^{\nu-k+1}}, \tag{26}$$

where

$$f_\nu(x) = \gamma_{0\nu}x^\nu + \gamma_{1\nu}x^{\nu-1} + \ldots + \gamma_{\nu\nu}$$

and C is a circle about the point $x=0$ as a center. From this it follows at once that

$$|\gamma_{k\nu}|\rho^{\nu-k} \leqq \frac{1}{2\pi\rho} M_\rho[f_\nu(x)], \tag{27}$$

where $M_\rho[f_\nu(x)]$ denotes the maximum value of $|f_\nu(x)|$ on the circle of radius ρ about the point $x=0$ as a center. There are large classes of cases in which it may readily be shown that relation (25) is implied by relation (27).

Let us consider the application of these results in the case of factorial series. Here $g(x) = 1/\Gamma(x)$; and we have

$$\frac{g(x)}{g(x+n)} = x(x+1)(x+2)\ldots(x+n).$$

It is obvious that the quantities γ_{sn} are all positive or zero, and that they are zero in case $s \geqq n$. Moreover,

$$\gamma_{0\nu}\rho^\nu + \gamma_{1\nu}\rho^{\nu-1} + \ldots + \gamma_{\nu\nu} = \frac{g(\rho)}{g(\rho+\nu)}.$$

Hence, conditions (23) and (24) are satisfied in case (18) converges for sufficiently large values of $|x|$, say for $|x| \geqq R$. Moreover, for this special case it is clear that we may take $\sigma_1=0$ and $\sigma=R-\mu$ and that μ now has the value $\frac{1}{2}$. Then it is easy to see that the line of absolute convergence of the series in (21), in this case cuts the axis of reals at a point not further to the right than a unit's distance to the right of the rightmost point of the circle of convergence of the power series in $1/x$ by which $f(x)$ may be represented.

Again, if we take for $g(x)$ the value

$$g(x)=\frac{1}{a^x\Gamma\left(x+\frac{b}{a}\right)}, \tag{28}$$

we have

$$\frac{g(x)}{g(x+1)}=ax+b.$$

Then, as in the preceding case, it is easy to see that the conditions of Theorem II are satisfied provided that (18) converges for sufficiently large $|x|$ and a is positive while b is positive or zero. If we put $b=0$, series (1) obviously takes the special form

$$S_1(x)=c_0+\sum_{n=1}^{\infty}\frac{c_n}{ax(ax+a)(ax+2a)\ldots.(ax+\overline{n-1}a)}.$$

Replacing ax by z, we may write this in the form

$$\overline{S}_a(z)=c_0+\sum_{n=1}^{\infty}\frac{c_n}{z(z+a)(z+2a)\ldots.(z+[n-1]a)}. \tag{29}$$

Series of this class play a leading rôle in Nörlund's fundamental paper on factorial series to which reference has already been made.

Again, if we put Mz for x and $\alpha+1$ for b/a in the function $g(x)$ of equation (28) the corresponding series $S(x)$ is transformed into the form

$$b_0+\frac{b_1}{Mz+\alpha+1}+\frac{b_2}{(Mz+\alpha+1)(Mz+\alpha+2)}$$
$$+\frac{b_3}{(Mz+\alpha+1)(Mz+\alpha+2)(Mz+\alpha+3)}+\ldots, \tag{30}$$

a series which plays the leading rôle in the important paper of Watson's referred to above. Watson exhibits a large class of functions expansible in converging series (30).

One may easily construct many other particular functions $g(x)$ satisfying those hypotheses of Theorem II which relate to $g(x)$ provided that the associated series (18) converges for sufficiently large $|x|$. Some of the most interesting of such functions $g(x)$ are readily expressible in terms of the gamma function. Such a one, for instance, is the function $g(x)$,

$$g(x)=\frac{\Gamma(x+1)}{\Gamma(x)\Gamma(x+3)}. \tag{31}$$

Here we have

$$\frac{g(x)}{g(x+1)}=\frac{x(x+3)}{x+1}.$$

Employing the relation

$$\frac{g(x)}{g(x+n)}=\frac{g(x)}{g(x+1)}\cdot\frac{g(x+1)}{g(x+2)}\ldots.\frac{g(x+n-1)}{g(x+n)},$$

it is thence easy to see that $g(x)/g(x+n)$ is a polynomial with non-negative real coefficients provided that n is sufficiently large. Hence, relations (23) and (24) are satisfied.

In a similar way one may treat the function

$$g(x)=\frac{\Gamma(x+1)\Gamma(x+3)}{\Gamma(x)\Gamma(x+2)\Gamma(x+4)}. \tag{32}$$

It is clear that one may thus form an indefinitely great number of similar functions $g(x)$ such that in each case $g(x)/g(x+1)$ is a rational function of x, and $g(x)/g(x+n)$ is a polynomial with non-negative real coefficients provided that n is sufficiently large.

§ 5. *Representation of Given Functions in the Form of Convergent Series $S(x)$.*

By means of Borel's method of summation of series (in general divergent) we shall now show how to represent functions of a certain important class in the form of convergent series $S(x)$. The functions treated are those obtained by taking the sum of a series of the form

$$\beta_0+\frac{\beta_1}{x}+\frac{\beta_2}{x^2}+\frac{\beta_3}{x^2}+\ldots. \tag{33}$$

by means of Borel's integral method of summation. Denoting the Borel integral sum of the series in (33) by $f(x)$, we have by definition:*

$$f(x)=\int_0^\infty xe^{-tx}\phi(t)\,dt, \tag{34}$$

where

$$\phi(t)=\beta_0+\beta_1 t+\frac{\beta_2 t^2}{2\,!}+\frac{\beta_3 t^3}{3\,!}+\ldots+\frac{\beta_n t^n}{n\,!}+\ldots. \tag{35}$$

We assume† that the series in (35) is convergent for all values of t, and that the function $\phi(t)$ defined by it is such that the integral in (34) exists. We say then that the series (34) is summable to the sum $f(x)$.

In this section we shall suppose† also that the asymptotic series in (3) is summable in the sense of Borel, so that we have

$$\frac{g(x+n)}{g(x)}=\int_0^\infty xe^{-tx}\phi_n(t)\,dt, \qquad n=1,2,3,\ldots, \tag{36}$$

where

$$\phi_n(t)=\frac{\beta_{0n}t^n}{n\,!}+\frac{\beta_{1n}t^{n+1}}{(n+1)\,!}+\frac{\beta_{2n}t^{n+2}}{(n+2)\,!}+\ldots. \tag{37}$$

The series in (37) we shall take to be convergent for all values of t.

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* Borel, "Leçons sur les séries divergentes," p. 108 ff. We have made certain obvious reductions so as to obtain the form convenient to use with descending series (33) rather than with ascending series.

† The restrictions made here are stronger than are essential to the argument. They may be weakened in accordance with certain general considerations mentioned by Borel (*loc. cit.*, p. 99).

Of the function $f(x)$ defined in (34) it is easy to obtain a *formal* expansion in series $S(x)$. For this purpose we note that by means of equation (9) it is easy to establish the formal relation

$$\beta_0+\beta_1 t+\ldots\ldots+\frac{\beta_n t^n}{n!}+\ldots\ldots=\sum_{n=0}^{\infty} c_n \sum_{\nu=n}^{\infty} \beta_{\nu-n,n}\frac{t^\nu}{\nu!}\equiv\sum_{n=0}^{\infty} c_n \phi_n(t). \qquad (38)$$

Replacing $\phi(t)$ in (34) by the second member of the last equation we have the formal relation

$$f(x)=\int_0^\infty\left(\sum_{n=0}^{\infty} c_n x e^{-tx}\phi_n(t)\right)dt. \qquad (39)$$

If, still proceeding formally, we integrate term by term the series denoted by the summation in (39) and make use of (36), we have

$$f(x)=\sum_{n=0}^{\infty} c_n \frac{g(x+n)}{g(x)}. \qquad (40)$$

Hence,

THEOREM III. *In all cases in which* (36) *and* (38) *are valid relations, and the series denoted by the outer summation in* (39) *is term by term integrable from zero to infinity, the Borel sum* $f(x)$ *of series* (33) *is represented in* (40) *in the form of a converging series* $S(x)$.

This result is capable of a ready generalization as follows: From (34) we have

$$f(ax)=\int_0^\infty axe^{-atx}\phi(t)\,dt=\int_0^\infty xe^{-tx}\psi(t)\,dt, \qquad (41)$$

where

$$\psi(t)=\beta_0+\frac{\beta_1}{a}t+\frac{\beta_2}{a^2}t^2+\frac{\beta_3}{a^3}t^3+\ldots\ldots$$

Analogous to (9) we now form the equations

$$\frac{\beta_\nu}{a^\nu}=\bar{c}_0\beta_{\nu 0}+\bar{c}_1\beta_{\nu-1,1}+\bar{c}_2\beta_{\nu-2,2}+\ldots\ldots+\bar{c}_\nu\beta_{0\nu}, \qquad \nu=0,1,2,\ldots\ldots,$$

thus introducing the quantities $\bar{c}_0, \bar{c}_1, \bar{c}_2, \ldots\ldots$ Thence proceeding formally, we have

$$\beta_0+\frac{\beta_1}{a}t+\frac{\beta_2}{a^2}t^2+\ldots\ldots=\sum_{n=0}^{\infty}\bar{c}_n\sum_{\nu=n}^{\infty}\beta_{\nu-n,n}\frac{t^\nu}{\nu!}\equiv\sum_{n=0}^{\infty}\bar{c}_n\phi_n(t); \qquad (42)$$

whence, as before, we obtain

$$f(ax)=\int_0^\infty\left(\sum_{n=0}^{\infty}\bar{c}_n xe^{-tx}\phi_n(t)\right)dt. \qquad (43)$$

Integrating term by term, we have

$$f(ax)=\sum_{n=0}^{\infty}\bar{c}_n\frac{g(x+n)}{g(x)};$$

or

$$f(x)=\sum_{n=0}^{\infty}\bar{c}_n\frac{g(x/a+n)}{g(x/a)}. \tag{44}$$

We are thus lead to the following result:

THEOREM IV. *In all cases in which* (36) *and* (42) *are valid relations and the series denoted by the summation in* (43) *is term by term integrable from zero to infinity, the Borel sum* $f(x)$ *of series* (33) *is represented in* (44) *in the form of a converging series into which a series* $S(x)$ *is readily transformed.*

If we take for $g(x)$ the particular value

$$g(x)=\frac{1}{a^x\Gamma(x)},$$

then relation (44) takes the special form

$$f(x)=\bar{c}_0+\frac{\bar{c}_1}{x}+\frac{\bar{c}_2}{x(x+a)}+\frac{\bar{c}_3}{x(x+a)(x+2a)} + \frac{\bar{c}_4}{x(x+a)(x+2a)(x+3a)}+\cdots\cdots \tag{45}$$

That is to say, the Borel sum $f(x)$ of (33) is always represented formally by the series in (45). In case (33) is absolutely and uniformly summable by the integral method of Borel to the sum $f(x)$, then Nörlund (*loc. cit.*, p. 379) has shown that $f(x)$ has an actual convergent expansion (45) in case a is a sufficiently large positive number dependent upon $f(x)$. Hence the formal result in (45), and therefore the more general one in (44), has a wide range of useful applicability.

For special functions $g(x)$ (of which that in the preceding paragraph is an example) and corresponding classes of functions $f(x)$ it is possible, as we have just seen, to state more explicit and precise results than those obtained in Theorems III and IV; but it seems to be difficult to render these theorems more precise without further restrictions on $g(x)$. I propose as an important problem the determination of classes of functions $g(x)$, and corresponding classes of functions $f(x)$, such that the representations (40) and (44) are valid, either separately or simultaneously.

UNIVERSITY OF ILLINOIS, *February*, 1917.

Transformations of Planar Nets.

By Luther Pfahler Eisenhart.

1. **Introduction.** When a surface is referred to a conjugate system of curves, u and v being the parameters, the four homogeneous point coordinates, $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}$, are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u \partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} + c\theta, \tag{1}$$

where a, b, c are functions of u and v. We say that the parametric curves u=const., v=const. form a net $N(x)$, x indicating any of the four coordinates, and call (1) *the point equation* of $N(x)$.

If θ is any other solution of (1), the equations

$$\frac{\partial x_1}{\partial u} = h\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right), \quad \frac{\partial x_1}{\partial v} = l\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right), \tag{2}$$

satisfy the condition of integrability, provided that h and l are any pair of solutions * of

$$\frac{\partial h}{\partial v} = \left(\frac{\partial \log\theta}{\partial v} - a\right)(h-l), \quad \frac{\partial l}{\partial u} = \left(\frac{\partial \log\theta}{\partial u} - b\right)(l-h). \tag{3}$$

Moreover, the four functions x_1 obtained by quadratures from (2) when x takes on the four values of the coordinates of $N(x)$ are solutions of the same equation

$$\frac{\partial^2\theta_1}{\partial u \partial v} + \frac{l}{h}\left(\frac{\partial \log\theta}{\partial v} - a\right)\frac{\partial\theta_1}{\partial u} + \frac{h}{l}\left(\frac{\partial \log\theta}{\partial u} - b\right)\frac{\partial\theta_1}{\partial v} = 0. \tag{4}$$

Hence these four quantities x_1 are the homogeneous point coordinates of a net $N_1(x_1)$.

The points F_1 and F_2, with respective homogeneous coordinates

$$\theta x_1 - hx, \quad \theta x_1 - lx, \tag{5}$$

lie on the line joining corresponding points, M and M_1, of the nets N and N_1, and are such that as u or v varies, F_1 or F_2, respectively, moves tangentially

* We exclude the case $h=l=$const., since in this case the two surfaces are homothetic with respect to the origin.

to the line. Hence the ruled surfaces u=const., v=const. of the congruence G of lines MM_1 are its developables. We say that the nets N and N_1 are *conjugate* to the congruence G, since the developables of the latter meet the surfaces on which N and N_1 lie in these nets. Two nets so related to a congruence are said to be in the relation of a *transformation* T with one another. Conversely, any two nets so related can be defined analytically by (2) and (3), as we have shown previously.*

In § 2 we show that if two nets N and N_1 are in the relation of a transformation T, the same is true of their first Laplace transforms, and likewise their minus first Laplace transforms. This result holds for nets in space of any order.

Any three of the functions x are the homogeneous coordinates of the planar net, say $P(x)$, into which $N(x)$ is projected from a suitable vertex of the coordinate tetrahedron upon the opposite face. Hence the preceding results can be applied to the transformation of planar nets. The lines of the congruence G project into lines of the plane, and thus the developables of the congruence lose their significance, and these transformations in general are not of much interest. However, there are certain types of planar nets characterized, for example, by geometric properties. The above general method can be used to obtain transformations of nets of a type into nets of the same type.

We consider transformations K of planar nets whose equation (1) has equal invariants into nets of the same kind. In a previous paper† we have shown how one can determine by quadratures a surface S whose asymptotic curves project into any given planar net with equal invariants. We show in § 4 that a transformation K of the planar net is equivalent to the determination of a surface S_1 such that S and S_1 are the focal surfaces of a W-congruence. We make use of these results in § 5 to determine the equations of a W-congruence.

There are certain planar nets with equal invariants which are reproduced after three transformations of Laplace. We refer to them as *nets of period* 3. They are of three types. The remainder of the paper is devoted to a study of the transformations K of these nets into nets of the same kind. A theorem of permutability of these transformations is established. We discover also another transformation of nets of period 3, purely analytic in character, and find that their determination and that of transformations K reduce to equivalent analytical problems.

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* *Transactions of the American Mathematical Society*, Vol. XVIII (1917), pp. 97–124.

† *Annals of Mathematics*, Series 2, Vol. XVIII (1917), pp. 221–225.

In § 12 we show that the preceding results can be interpreted as giving transformations of certain surfaces discovered by Tzitzeica.

2. **Laplace Transformations and Transformations *T*.** The tangents to the curves $v=\text{const.}$ of a net N are tangent to the curves $u=\text{const.}$ of a net $(N)_1$, and the tangents to the curves $u=\text{const.}$ of N are tangent to the curves $v=\text{const.}$ of a net $(N)_{-1}$. The nets $(N)_1$ and $(N)_{-1}$ are called the *first* and *minus first Laplace transforms* of N. Their respective homogeneous coordinates y and z can be given the forms

$$y=\frac{\partial x}{\partial u}-bx, \quad z=\frac{\partial x}{\partial v}-ax.^*$$

We shall prove the theorem:

If N_1 is a T transform of N determined by a function θ, the first and minus first Laplace transforms of N_1 are T transforms of the first and minus first Laplace transforms of N by means of the respective functions

$$\frac{\partial\theta}{\partial u}-b\theta, \quad \frac{\partial\theta}{\partial v}-a\theta.$$

In consequence of (4) it follows that the homogeneous coordinates of the first Laplace transform of N_1 are of the form

$$\frac{\partial x_1}{\partial u}+\frac{h}{l}\left(\frac{\partial \log\theta}{\partial u}-b\right)x_1.$$

By means of (2) this is reducible to such a form that we can take for the coordinates y_1 of this transform the expressions

$$y_1=\frac{l\dfrac{\partial}{\partial u}\left(\dfrac{x}{\theta}\right)}{\dfrac{\partial}{\partial u}\log\theta-b}+x_1.$$

Making use of the fact that x and θ are solutions of equation (1), we obtain

$$\frac{\partial y_1}{\partial u}=\bar{h}\frac{\partial}{\partial u}\left(\frac{y}{\bar\theta}\right), \quad \frac{\partial y_1}{\partial v}=\bar{l}\frac{\partial}{\partial v}\left(\frac{y}{\bar\theta}\right),$$

where

$$\bar\theta=\frac{\partial\theta}{\partial u}-b\theta, \quad \bar{h}=\frac{\dfrac{\partial\log\theta}{\partial u}-b}{k}+l,$$

* Darboux, *Leçons*, Vol. II, pp. 27, 28.

k being the invariant of equation (1) defined by

$$k=\frac{\partial b}{\partial v}-ab-c.$$

Since the above equations necessarily satisfy the conditions of integrability, we have proved that the two nets are in the relation of a transformation T. By similar means we show that the minus first transforms are so related.

3. **Equations of Transformations T.** The three homogeneous coordinates $x^{(1)}, x^{(2)}, x^{(3)}$ of a planar net $P(x)$ are solutions not only of equation (1), but also of two equations of the form

$$\frac{\partial^2\theta}{\partial u^2}=a'\frac{\partial\theta}{\partial u}+b'\frac{\partial\theta}{\partial v}+c'\theta, \quad \frac{\partial^2\theta}{\partial v^2}=a''\frac{\partial\theta}{\partial u}+b''\frac{\partial\theta}{\partial v}+c''\theta, \tag{6}$$

where the coefficients are functions of u and v, which must be such that the conditions of integrability

$$\frac{\partial}{\partial v}\left(\frac{\partial^2 x}{\partial u^2}\right)=\frac{\partial}{\partial u}\left(\frac{\partial^2 x}{\partial u\partial v}\right), \quad \frac{\partial}{\partial u}\left(\frac{\partial^2 x}{\partial v^2}\right)=\frac{\partial}{\partial v}\left(\frac{\partial^2 x}{\partial u\partial v}\right),$$

are satisfied by all three functions x. This leads to the following six conditions:

$$\left.\begin{array}{ll}
\dfrac{\partial a'}{\partial v}+b'a''=\dfrac{\partial a}{\partial u}+ab+c, & \dfrac{\partial b'}{\partial v}+a'b+b'b''+c'=\dfrac{\partial b}{\partial u}+ab'+b^2, \\
\dfrac{\partial c'}{\partial v}+a'c+b'c''=\dfrac{\partial c}{\partial u}+ac'+bc, & \dfrac{\partial b''}{\partial u}+b'a''=\dfrac{\partial b}{\partial v}+ab+c, \\
\dfrac{\partial a''}{\partial u}+a'a''+ab''+c''=\dfrac{\partial a}{\partial v}+a''b+a^2, & \dfrac{\partial c''}{\partial u}+a''c'+b''c=\dfrac{\partial c}{\partial v}+ac+bc''.
\end{array}\right\} \tag{7}$$

By making use of equations (6), we are able to determine functions k, m, n so that, for all three x's,

$$x_1=kx+m\frac{\partial x}{\partial u}+n\frac{\partial x}{\partial v}. \tag{8}$$

In fact, when this expression is substituted in (2), we find that k, m, n must satisfy the completely integrable system of equations

$$\left.\begin{array}{ll}
\dfrac{\partial k}{\partial u}+mc'+nc+\dfrac{h}{\theta^2}\dfrac{\partial\theta}{\partial u}=0, & \dfrac{\partial m}{\partial u}+k+ma'+na-\dfrac{h}{\theta}=0, \\
\dfrac{\partial n}{\partial u}+mb'+nb=0, & \dfrac{\partial k}{\partial v}+mc+nc''+\dfrac{l}{\theta^2}\dfrac{\partial\theta}{\partial v}=0, \\
\dfrac{\partial m}{\partial v}+ma+na''=0, & \dfrac{\partial n}{\partial v}+k+mb+nb''-\dfrac{l}{\theta}=0.
\end{array}\right\} \tag{9}$$

4. **Planar Nets with Equal Invariants.** When $P(x)$ and $P_1(x_1)$ are the projections of two nets $N(x)$ and $N_1(x_1)$ in the relation of a transformation T, the points of the plane whose corresponding coordinates are given by (5) are points of contact of the line $\overline{PP_1}$ with its envelope as u or v varies. We consider the case when these points are harmonic to P and P_1, and say that P and P_1 are in the relation of a *transformation K*. From (5) it follows that we must have $l=-h$, and consequently from (3) that

$$\frac{\partial a}{\partial u}=\frac{\partial b}{\partial v},$$

that is, equation (1) has equal invariants.* By making a suitable choice of the homogeneous coordinates x in this case we can take equation (1) in the form

$$\frac{\partial^2\theta}{\partial u\partial v}=c\theta. \tag{10}$$

Now, in all generality we have from (3)

$$h=-l=-\theta^2, \tag{11}$$

so that, if x_1 in (2) be replaced by $x_1\theta$, we have

$$\frac{\partial}{\partial u}(x_1\theta)=-\theta\frac{\partial x}{\partial u}+x\frac{\partial\theta}{\partial u},\quad \frac{\partial}{\partial v}(x_1\theta)=\theta\frac{\partial x}{\partial v}-x\frac{\partial\theta}{\partial v}, \tag{12}$$

and the point equation of N_1 is

$$\frac{\partial^2\theta_1}{\partial u\partial v}=\theta\frac{\partial^2}{\partial u\partial v}\frac{1}{\theta}\cdot\theta_1. \tag{13}$$

We have shown † that if we define three functions $y^{(1)}$, $y^{(2)}$, $y^{(3)}$ by

$$\frac{\partial y^{(1)}}{\partial u}=x^{(2)}\frac{\partial x^{(3)}}{\partial u}-x^{(3)}\frac{\partial x^{(2)}}{\partial u},\quad \frac{\partial y^{(1)}}{\partial v}=-x^{(2)}\frac{\partial x^{(3)}}{\partial v}+x^{(3)}\frac{\partial x^{(2)}}{\partial v}, \tag{14}$$

and by similar equations obtained by permuting the hyperscripts cyclically, the function $x^{(4)}$ given by

$$x^{(4)}=x^{(1)}y^{(1)}+x^{(2)}y^{(2)}+x^{(3)}y^{(3)}, \tag{15}$$

and $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are the homogeneous point coordinates of a surface S, upon which the parametric curves are the asymptotic lines. These lines and the planar net are in perspective relation from the point (0, 0, 0, 1).

* This is in accordance with the similar result for nets in space first proved by Koenig's *Comptes Rendus*, Vol. CXIII (1891), p. 1022.

† *Annals, loc. cit.* The letters x and y must be interchanged to give the equations in use in the present paper.

Since $x^{(1)}, x^{(2)}, x^{(3)}$ are solutions of (10), equations (14) are of the form of equations of Lelieuvre[*] $y^{(1)}, y^{(2)}, y^{(3)}, 1$, are the point coordinates of a surface Σ referred to its asymptotic lines. Moreover, from (15) it is seen that Σ and S are polar reciprocal with respect to the quadric

$$Z^{(1)2}+Z^{(2)2}+Z^{(3)2}=Z^{(4)2}. \tag{16}$$

By means of the three functions $x_1^{(1)}, x_1^{(2)}\ x_1^{(3)}$ obtained from the quadratures (12), we define three functions $y_1^{(1)}, y_1^{(2)}, y_1^{(3)}$ by

$$\frac{\partial y_1^{(1)}}{\partial u}=x_1^{(2)}\frac{\partial x_1^{(3)}}{\partial u}-x_1^{(3)}\frac{\partial x_1^{(2)}}{\partial u}, \quad \frac{\partial y_1^{(1)}}{\partial v}=-x_1^{(2)}\frac{\partial x_1^{(3)}}{\partial v}+x_1^{(3)}\frac{\partial x_1^{(2)}}{\partial v}, \tag{17}$$

and similar equations obtained by permuting the hyperscripts cyclically. Now, as seen above, the functions $y_1^{(1)}, y_1^{(2)}, y_1^{(3)}, 1$, are the point coordinates of a surface Σ_1, referred to its asymptotic lines. Moreover, it is readily found† that

$$y_1^{(1)}=y^{(1)}+x^{(2)}x_1^{(3)}-x^{(3)}x_1^{(2)}. \tag{18}$$

From (14) and (15) it follows that the equation of the tangent plane to Σ is

$$x^{(1)}z^{(1)}+x^{(2)}z^{(2)}+x^{(3)}z^{(3)}-x^{(3)}=0,$$

the z's being current coordinates. It is readily seen that this plane passes through the point $(y_1^{(1)}, y_1^{(2)}, y_1^{(3)}, 1)$ of Σ_1. In like manner we show that the tangent plane to Σ_1 passes through the corresponding point on Σ. Hence Σ and Σ_1 are the focal surfaces of the congruence of lines joining corresponding points of these surfaces, which accordingly is a W-congruence.

By means of $y_1^{(1)}, y_1^{(2)}, y_1^{(3)}$ we obtain, as by (15), the surface S_1 whose asymptotic curves project from $(0, 0, 0, 1)$ into the net $P_1(x_1)$. Since S_1 is the polar transform of Σ_1 with respect to the quadric (16), and this transformation changes a W-congruence into a W-congruence, we have the theorem:

When two planar nets with equal invariants are in the relation of a transformation K, the two surfaces, whose asymptotic lines are the perspectives of the curves of the nets from a point, are the focal surfaces of a W-congruence.

5. **Equations of a *W*-congruence.** It is well known that the direction-cosines of the normal to a surface referred to its asymptotic lines are solutions of the same equation of Laplace with equal invariants.‡ It is possible to choose a suitable factor so that the direction-parameters (that is, quantities proportional to the direction-cosines) shall satisfy by an equation of the form (10).

* E., p. 193. A reference of this sort is to the author's "Differential Geometry."
† E., p. 418.
‡ E., p. 194.

In this case we say that the parameters are in the *normal form.* From (14) it is seen that $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are the direction-parameters of Σ in the normal form.

From (14) we have, by differentiation and reduction, with the aid of (6),

$$\frac{\partial^2 y^{(1)}}{\partial u^2}=a'\frac{\partial y^{(1)}}{\partial u}-b'\frac{\partial y^{(1)}}{\partial v},\quad \frac{\partial^2 y^{(1)}}{\partial v^2}=-a''\frac{\partial y^{(1)}}{\partial u}+b''\frac{\partial y^{(1)}}{\partial v},\tag{19}$$

and

$$\frac{\partial^2 y^{(1)}}{\partial u\partial v}=\frac{\partial \log x^{(3)}}{\partial v}\frac{\partial y^{(1)}}{\partial u}+\frac{\partial \log x^{(3)}}{\partial u}\frac{\partial y^{(1)}}{\partial v}.\tag{20}$$

The coordinates $y^{(2)}$, $y^{(3)}$ also satisfy (19), and $y^{(2)}$ satisfies (20). Hence we have the theorem:

When the asymptotic lines on a surface are projected orthogonally on a plane, taken as a coordinate plane, the equation satisfied by the cartesian coordinates of this planar net is (20), *where* $x^{(3)}$ *is the direction-parameter in the normal form of the normal to the surface with respect to the normal to the plane.*

When two surfaces Σ and Σ_1 are the focal surfaces of a W-congruence, we say that they are in the relation of a *W-transformation.* Equations (18) define the general transformation. They can be given another form by means of equations (8) for the case of transformations K. In accordance with § 3 we replace x_1 and k by $x_1\theta$ and $k\theta$, respectively, and equation (8) becomes

$$x_1=kx+\frac{m}{\theta}\frac{\partial x}{\partial u}+\frac{n}{\theta}\frac{\partial x}{\partial v}.\tag{21}$$

Since now $a=b=0$ and equations (11) hold, equations (9) become

$$\left.\begin{aligned}
&\theta\frac{\partial k}{\partial u}+(k-1)\frac{\partial\theta}{\partial u}+mc'+nc=0,\quad \theta\frac{\partial k}{\partial v}+(k+1)\frac{\partial\theta}{\partial v}+mc+nc''=0,\\
&\frac{\partial m}{\partial u}+k\theta+ma'+\theta=0,\quad \frac{\partial m}{\partial v}+na''=0,\\
&\frac{\partial n}{\partial u}+mb'=0,\quad \frac{\partial n}{\partial v}+k\theta+nb''-\theta=0.
\end{aligned}\right\}\tag{22}$$

Substituting the values from (21) in (18), we get the desired equations of transformation, namely

$$y_1=y+\frac{m}{\theta}\frac{\partial y}{\partial u}-\frac{n}{\theta}\frac{\partial y}{\partial v}.\tag{23}$$

Hence the problem of W-transformations is equivalent to the solution of equation (10) and of the completely integrable system (22).

6. **Nets of Period 3.** We apply the preceding methods to nets for which $a=b=0$, $c'=c''=0$. These nets are characterized by the property that any one of them is reproduced after three transformations of Laplace. In this sense they are of period 3.

It follows from (7) that in the present case

$$a'=\frac{\partial \log c}{\partial u}, \quad b''=\frac{\partial \log c}{\partial v}, \quad b'=\frac{U}{c}, \quad a''=\frac{V}{c},$$

where U and V are arbitrary functions of u and v respectively. One or both of these functions can be taken equal to zero. If they are not zero, by a suitable choice of the parameters u and v of the nets we can take U and V equal to unity. Hence in all generality we have

$$b'=\frac{\varepsilon}{c}, \quad a''=\frac{\eta}{c},$$

where ε and η take the values 0 or 1, and the differential equations of the net are

$$\frac{\partial^2 x}{\partial u^2}=\frac{\partial \log c}{\partial u}\frac{\partial x}{\partial u}+\frac{\varepsilon}{c}\frac{\partial x}{\partial v}, \quad \frac{\partial^2 x}{\partial u \partial v}=cx, \quad \frac{\partial^2 x}{\partial v^2}=\frac{\eta}{c}\frac{\partial x}{\partial u}+\frac{\partial \log c}{\partial v}\frac{\partial x}{\partial v}. \tag{24}$$

All the conditions (7) are satisfied provided that c satisfies

$$\frac{\partial^2}{\partial u \partial v}\log c=c-\frac{\varepsilon\eta}{c^2}. \tag{25}$$

For the present case equations (22) reduce to

$$\left.\begin{aligned}
&\theta\frac{\partial k}{\partial u}+nc+(k-1)\frac{\partial \theta}{\partial u}=0, \quad \theta\frac{\partial k}{\partial v}+mc+(k+1)\frac{\partial \theta}{\partial v}=0,\\
&\frac{\partial m}{\partial u}+k\theta+m\frac{\partial \log c}{\partial u}+\theta=0, \quad \frac{\partial m}{\partial v}+\frac{n\eta}{c}=0,\\
&\frac{\partial n}{\partial u}+\frac{m\varepsilon}{c}=0, \quad \frac{\partial n}{\partial v}+k\theta+n\frac{\partial \log c}{\partial v}-\theta=0.
\end{aligned}\right\} \tag{26}$$

Eliminating k from the second and third of these equations, we get

$$\frac{\partial^2 m}{\partial u \partial v}+\frac{\partial \log c}{\partial u}\frac{\partial m}{\partial v}+\left(\frac{\partial^2 \log c}{\partial u \partial v}-c\right)m=0.$$

It is readily seen that $\frac{\partial \theta}{\partial v}/c$ is a solution of this equation, so that we replace m by $\mu\frac{\partial \theta}{\partial v}/c$. Similarly, we replace n by $\nu\frac{\partial \theta}{\partial u}/c$. Then (21) becomes

$$x_1=xk+\frac{\mu}{c\theta}\frac{\partial \theta}{\partial v}\frac{\partial x}{\partial u}+\frac{\nu}{c\theta}\frac{\partial \theta}{\partial u}\frac{\partial x}{\partial v}, \tag{27}$$

and (26) reduces to

$$\left.\begin{aligned}
&\frac{\partial k}{\partial u}+(k+\nu-1)\frac{1}{\theta}\frac{\partial\theta}{\partial u}=0, \quad \frac{\partial k}{\partial v}+(k+\mu+1)\frac{1}{\theta}\frac{\partial\theta}{\partial v}=0,\\
&\frac{\partial\mu}{\partial u}\frac{\partial\theta}{\partial v}+c\theta(k+\mu+1)=0, \quad \frac{\partial\mu}{\partial v}\frac{\partial\theta}{\partial u}+c\theta(k+\nu-1)=0,\\
&\frac{\partial\nu}{\partial u}\frac{\partial\theta}{\partial u}+\nu\frac{\partial^2\theta}{\partial u^2}-\nu\frac{\partial\theta}{\partial u}\frac{\partial\log c}{\partial u}+\frac{\varepsilon\mu}{c}\frac{\partial\theta}{\partial v}=0,\\
&\frac{\partial\mu}{\partial v}\frac{\partial\theta}{\partial v}+\mu\frac{\partial^2\theta}{\partial v^2}-\mu\frac{\partial\theta}{\partial v}\frac{\partial\log c}{\partial v}+\frac{\eta\mu}{c}\frac{\partial\theta}{\partial u}=0.
\end{aligned}\right\} \quad (28)$$

If the net $N_1(x_1)$ is to be of the same kind, we must have

$$\frac{\partial^2 x_1}{\partial u^2}=\frac{\partial\log c_1}{\partial u}\frac{\partial x_1}{\partial u}+\frac{U}{c_1}\frac{\partial x_1}{\partial v}, \quad \frac{\partial^2 x_1}{\partial u\partial v}=c_1x_1, \quad \frac{\partial^2 x_1}{\partial v^2}=\frac{V}{c_1}\frac{\partial x_1}{\partial u}+\frac{\partial\log c_1}{\partial v}\frac{\partial x_1}{\partial v}, \quad (29)$$

where, as follows from (13), c_1 is given by

$$c_1=\theta\frac{\partial^2}{\partial u\partial v}\frac{1}{\theta}=-c+\frac{2}{\theta^2}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}. \quad (30)$$

When the expression for x_1, as given by (27), is differentiated and substituted in the first and third of (29), the resulting equations are reducible by means of (24) and (28) to equations of the form

$$Ax+B\frac{\partial x}{\partial u}+C\frac{\partial x}{\partial v}=0, \quad A_1x+B_1\frac{\partial x}{\partial u}+C_1\frac{\partial x}{\partial v}=0,$$

where A, B, C, A_1, B_1, C_1 are determinate functions of u and v. As the above equations must be satisfied by $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$, the coefficients must vanish identically. Hence we have the following equations of condition:

$$\left.\begin{aligned}
&U(k+\mu+1)=0, \quad V(k+\nu-1)=0,\\
&U\left(\frac{2\nu}{c\theta^2(1-k)}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}-1\right)+\varepsilon\left(1-\frac{2}{c\theta^2}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}\right)=0,\\
&V\left(\frac{2\mu}{c\theta^2(1+k)}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}+1\right)-\eta\left(1-\frac{2}{c\theta^2}\frac{\partial\theta}{\partial u}\frac{\partial\theta}{\partial v}\right)=0,\\
&\frac{\partial^2\theta}{\partial u^2}=\frac{\partial\log c}{\partial u}\frac{\partial\theta}{\partial u}+\frac{1+k}{1-k}\frac{U}{c}\frac{\partial\theta}{\partial v}, \quad \frac{\partial^2\theta}{\partial v^2}=\frac{1-k}{1+k}\frac{V}{c}\frac{\partial\theta}{\partial u}-+\frac{\partial\log c}{\partial v}\frac{\partial\theta}{\partial v}.
\end{aligned}\right\} \quad (31)$$

We break up the discussion of the problem into three cases.

7. **When $U\neq 0$, $V\neq 0$. Nets of the First Type.** Comparing equations (28) and (31), we find that k, μ and ν are constants such that

$$k+\mu+1=0, \quad k+\nu-1=0. \quad (32)$$

Moreover, we have

$$U=\varepsilon=1, \quad V=\eta=1.$$

If k and x_1 be replaced by $1/e$ and x_1/e, equation (27) becomes

$$x_1=x-\frac{1+e}{c\theta}\frac{\partial\theta}{\partial v}\frac{\partial x}{\partial u}-\frac{1-e}{c\theta}\frac{\partial\theta}{\partial u}\frac{\partial x}{\partial v}. \tag{33}$$

The equations of condition (28) and (31) are satisfied, provided θ is a solution of the system

$$\left.\begin{aligned}&\frac{\partial^2\theta}{\partial u^2}=\frac{\partial\log c}{\partial u}\frac{\partial\theta}{\partial u}+\frac{e+1}{e-1}\frac{\varepsilon}{c}\frac{\partial\theta}{\partial v}, \quad \frac{\partial^2\theta}{\partial u\partial v}=c\theta,\\ &\frac{\partial^2\theta}{\partial v^2}=\frac{e-1}{e+1}\frac{\eta}{c}\frac{\partial\theta}{\partial u}+\frac{\partial\log c}{\partial v}\frac{\partial\theta}{\partial v},\end{aligned}\right\} \tag{34}$$

with $\varepsilon=\eta=1$. We say that a net with equations (24) where $\varepsilon=\eta=1$ is of the *first type.*

The general solution of equations (34) involves three arbitrary constants in addition to e. Hence we have the theorem:

A planar net of period 3 of the first type admits ∞^4 transforms of the same kind.

We remark that equations (29) for N_1 are of the same form as (24).

From the results of § 2 it follows that the other two nets forming with N a closed cycle under Laplace transformations are in the relation of transformations K with the two nets forming with N_1 a closed cycle. This remark applies equally well to nets of the second and third types to be discussed in the following sections.

8. **When $U=0$, $V\neq 0$. Nets of the Second Type.** From the second of (31) it follows that $\varepsilon=0$ or $c_1=0$. Assuming $\varepsilon=0$, we find, as in the previous case, that k, μ and ν are constants in the relations (32); also that $V=\eta=1$. Now (25) becomes

$$\frac{\partial^2\log c}{\partial u\partial v}=c, \tag{35}$$

of which the general integral is

$$c=-\frac{2U'V'}{(1+UV)^2}, \tag{36}$$

where U and V are arbitrary functions of u and v respectively, and the primes indicate differentiation with respect to the argument. When $\varepsilon=0$, $\eta=1$, the net is of the *second type.*

When the value (36) of c is substituted in the second of (34), we find the general integral of the resulting equation; it is

$$\theta=\frac{U_1'}{U'}+\frac{V_1'}{V'}-2\frac{U_1V+V_1U}{1+UV'}, \tag{37}$$

where U_1 and V_1 are arbitrary functions of u and v respectively.

The fifth of (31) is $\frac{\partial^2\theta}{\partial u^2}=\frac{\partial \log c}{\partial u}\frac{\partial\theta}{\partial u}$, of which the general integral is

$$\frac{\partial\theta}{\partial u}=c\phi(v),$$

ϕ being an arbitrary function of v. When the value (37) of θ is substituted in this equation, it becomes

$$\frac{1}{U'}\left(\frac{U_1'}{U'}\right)'(1+UV)^2-2V\frac{U_1'}{U'}(1+UV)+2(U_1V^2-V_1)+2\phi(v)V'=0. \tag{38}$$

Differentiating with respect to u, we get

$$\left[\frac{1}{U'}\left(\frac{U_1'}{U'}\right)'\right]'(1+UV)^2=0.$$

Hence

$$U_1=aU^2+bU+c, \tag{39}$$

where a, b, c are constants. Substituting in (38), we find

$$a-bV+cV^2=V_1-\phi V'.$$

If we replace $\phi V'$ by V_2, so that

$$V_1=a-bV+cV^2+V_2, \tag{40}$$

equation (37) reduces to

$$\theta=\frac{V_2'}{V'}-\frac{2UV_2}{1+UV}. \tag{41}$$

The last of (31) remains to be satisfied; it may be taken in the form of the last of (34). Substituting the expression (41) for θ in this equation, we have for the determination of V_2 the equation

$$V'\left(\frac{V_2'}{V'}\right)''-V''\left(\frac{V_2'}{V'}\right)'-\frac{e-1}{e+1}V_2=0. \tag{42}$$

Thus the solution of this equation determines the transformations of nets of the second type into nets of the same kind. As before we have the theorem:

A planar net of the second type admits ∞^4 transforms of the same kind, and their determination requires the integration of an ordinary differential equation of the third order.

It remains for us to consider the second possibility for the satisfaction of the second of (31), namely, $\varepsilon=1$, $c_1=0$. Then, from (31) we get the two equa-

tions (32), and from the first two of (28) it follows that k, μ and ν are constants. Then the fifth equations of (28) and (31) are inconsistent.

From the results of §§ 6, 7 it follows that a net of the first or second type does not admit a transformation K into a net of the second or first type respectively.

So far as the actual determination of nets of the second type goes, it will be shown that it is a problem of the integration of an ordinary differential equation of the third order involving an arbitrary function. In fact, when in (1) and (6)

$$a=b=b'=c'=c''=0,$$

equations (7) are equivalent to

$$a'=\frac{\partial \log c}{\partial u}, \quad b''=\frac{\partial \log c}{\partial v}, \quad a''=\frac{2V_1}{c}, \quad \frac{\partial^2 \log c}{\partial u \partial v}=c,$$

where V_1 is an arbitrary function of v alone. The function c is given by (36). If U and V be taken as the parameters and be replaced by u and v, in place of (24) we have the system

$$\frac{\partial^2 x}{\partial u^2}=\frac{-2v}{1+uv}\frac{\partial x}{\partial u}, \quad \frac{\partial^2 x}{\partial u \partial v}=\frac{-2}{(1+uv)^2}x, \quad \frac{\partial^2 x}{\partial v^2}=V_1(1+uv)^2\frac{\partial x}{\partial u}-\frac{2u}{1+uv}\frac{\partial x}{\partial v}.$$

The integral of the second of these equations is of the form (37) with U and V replaced by u and v. In order that the other two equations shall be satisfied, we find, similarly to (41), that the coordinates of the net are of the form

$$x^{(i)}=\psi_i(v)-\frac{2v\psi_i}{1+uv},$$

where the functions ψ_i are linearly independent solutions of the equation

$$\psi'''-2V_1\psi=0.$$

Thus the above statement has been proved.

9. **Transformations L of Nets of the First and Second Types.** The equation (25) is such that if $c(u, v)$ is a solution, so also is $c_m \equiv c(mu, v/m)$, where m is any constant. Hence with a net N of the first or second type which has equations of the form (24) in which c is known there is associated a net N_m whose equations are

$$\frac{\partial^2 x_m}{\partial u^2}=\frac{\partial}{\partial u}\log c_m \frac{\partial x_m}{\partial u}+\frac{\varepsilon}{c_m}\frac{\partial x_m}{\partial v}, \quad \frac{\partial^2 x_m}{\partial u \partial v}=c_m x_m, \quad \frac{\partial^2 x_m}{\partial v^2}=\frac{\eta}{c_m}\frac{\partial x_m}{\partial u}+\frac{\partial}{\partial v}\log c_m \frac{\partial x_m}{\partial v}.$$

As this transformation from N to N_m is suggestive of the so-called Lie transformation of pseudo-spherical surfaces,* we call it a *transformation* L_m.

* E., p. 289.

We return to the consideration of equations (34) and note that if we effect the change of variables given by

$$u=\sqrt[3]{\frac{e-1}{e+1}}u_1,\quad v=\sqrt[3]{\frac{e+1}{e-1}}v_1,$$

these equations reduce to

$$\frac{\partial^2\theta}{\partial u_1^2}=\frac{\partial}{\partial u_1}\log c\,\frac{\partial\theta}{\partial u_1}+\frac{\varepsilon}{c}\,\frac{\partial\theta}{\partial v_1},\quad \frac{\partial^2\theta}{\partial u_1\partial v_1}=c\theta,\quad \frac{\partial^2\theta}{\partial v_1^2}=\frac{\eta}{c}\,\frac{\partial\theta}{\partial u_1}+\frac{\partial}{\partial v_1}\log c\,\frac{\partial\theta}{\partial v_1}.$$

Comparing these equations with the preceding set, we see that the general solution of these equations, and, consequently, of (34), is given by

$$\theta=\sum_1^3 a_i x_m^{(i)}\left(\frac{u}{m},\, mv\right),$$

where a_i are constants, $x_m^{(i)}(u,v)$ are the coordinates of the net N_m and

$$m=\sqrt[3]{\frac{e-1}{e+1}}.$$

Hence we have the theorem:

The complete determination of transformations K of nets of the first and second types is the same analytical problem as the complete determination of transformations L.

10. **When $U=V=0$. Nets of the Third Type.** We consider first the case where $c_1\neq 0$. Then, as follows from (31), $\varepsilon=\eta=0$. In this case the net N is said to be of the *third type*. Now the last two of (31) reduce to (34) with $\varepsilon=\eta=0$, and c is given by (36). Hence θ is of the form (37) with U_1 given by (39) and V_1 by a similar expression, say

$$V_1=a_1V^2+b_1V+c_1.$$

Accordingly θ is reducible to the form

$$\theta=[f(1-UV)+gU+hV]/(1+UV),$$

where f, g, h are arbitrary constants whose form in terms of the constants $a_1, b_1, \ldots, c_1$ is unessential.

When $\varepsilon=\eta=0$, it is not necessary to make a special choice of parameters u and v so as to reduce the equations of the net to the form (24). As the parameters are consequently undetermined, we can assume that they are chosen so that the most general form of θ is

$$\theta=[f(1-uv)+gu+hv]/(1+uv). \tag{43}$$

From the last two of (28) we have

$$\mu=U_1,\quad \nu=V_1,$$

where U_1 and V_1 are functions of u and v respectively. The third and fourth of (28) reduce to

$$\theta(k+\mu+1)=\frac{U_1'}{2}(h-2fu-gu^2),\quad \theta(k+\nu-1)=\frac{V_1'}{2}(g-2fv-hv^2), \tag{44}$$

expressions which are consistent with the first two of (28).

If equations (44) be subtracted from one another, we get

$$\begin{aligned}&[f(1-uv)+gu+hv](U_1-V_1+2)\\ &\qquad=\left[\frac{U_1'}{2}(h-2fu-gu^2)-\frac{V_1'}{2}(g-2fv-hv^2)\right](1+uv).\end{aligned} \tag{45}$$

Differentiating with respect to u and v, we have

$$\begin{aligned}&[(h-fu)U_1]'-[(h-gv)V_1]'-2f\\ &\qquad=\left[\frac{uU_1'}{2}(h-2fu+gu^2)\right]'-\left[\frac{vV_1'}{2}(g-2fv+hv^2)\right]'.\end{aligned}$$

This equation may be replaced by the two

$$\left.\begin{aligned}\frac{U_1'}{2}(h-2fu+gu^2)&=\left(\frac{h}{u}-f\right)U_1-2f+\alpha+\frac{\beta}{u},\\ \frac{V_1'}{2}(g-2fv+hv^2)&=\left(\frac{g}{v}-f\right)V_1+\alpha+\frac{\gamma}{v},\end{aligned}\right\} \tag{46}$$

where α, β, γ are constants. Substituting in (45), we obtain

$$\left(2f+gu-\frac{h}{u}\right)U_1-\left(2f+hv-\frac{g}{v}\right)V_1+4f+2gu+2hv-(1+uv)\left(\frac{\beta}{u}-\frac{\gamma}{v}\right)=0.$$

This equation is equivalent to

$$\begin{aligned}(h-2fu-gu^2)U_1&=(4f+\delta)u+2gu^2-\beta+\gamma u^2,\\ (g-2fv-hv^2)V_1&=\delta v-2hv^2-\gamma+\beta v^2,\end{aligned}$$

where δ is a constant. These results are consistent with (46) when $\delta=-2\alpha$, in which case we have from (44) for the determination of k,

$$k[f(1-uv)+gu+hv]=(f-\alpha)(1-uv)+(g+\gamma)u+(\beta-h)v.$$

When the constants α, β, γ satisfy the conditions

$$(\beta/h)-2=\gamma/g=-\alpha/f,$$

and only in this case, U_1 and V_1 are constants. Then k is constant, and the equations of the transformation are of the form (33). As in the other two cases, we have the theorem:

A planar net of the third type admits ∞^4 transforms of the same kind, and they can be found without quadrature.

We consider now the case where $c_1=0$. From (30) it follows that

$$\theta=1/(U+V),$$

where U and V are functions of u and v respectively. When this value is substituted in the second of (34), we find

$$c=2U'V'/(U+V)^2.$$

In order that this value may satisfy (25), we must have $\varepsilon\eta=0$. The last two of equations (31) reduce to

$$\frac{\partial^2\theta}{\partial u^2}=\frac{\partial \log c}{\partial u}\frac{\partial\theta}{\partial u}, \quad \frac{\partial^2\theta}{\partial v^2}=\frac{\partial \log c}{\partial v}\frac{\partial\theta}{\partial v},$$

which are satisfied identically by the above values, whatever be U and V.

From the fifth and fourth of (28) we get

$$\nu=V_1, \quad k=(U+V)(V_1'/2V')-V_1+1,$$

where V_1 is an arbitrary function of v. From the second of (28) we have

$$\mu=\frac{(U+V)^2}{V'}\left(\frac{V_1'}{2V'}\right)'-\frac{V_1'}{V'}(U+V)+V_1-2.$$

These values satisfy the first and third of equations (28) identically, but in order that the last of (28) be satisfied, V_1 must be such that

$$\left(\frac{V_1'}{V'}\right)''-\frac{V''}{V'}\left(\frac{V_1'}{V'}\right)'+\eta\frac{V_1}{V'}=0.$$

The function k is constant only when V_1 is constant. Then from the above equation $\eta=0$.

The results just obtained show that nets of the second and third types admit transformation K into nets whose equations are given by (1) and (6) when

$$a=b=c=b'=c'=a''=c''=0.$$

11. **Theorem of Permutability.** In this section we establish a theorem of permutability of the transformations of nets of period 3.

We take two functions θ_i and two constants e_i $(i=1, 2)$ satisfying the systems analogous to (34), namely

$$\left.\begin{aligned} &\frac{\partial^2\theta_i}{\partial u^2}=\frac{\partial \log c}{\partial u}\frac{\partial\theta_i}{\partial u}+\frac{e_i+1}{e_i-1}\frac{\varepsilon_i}{c}\frac{\partial\theta_i}{\partial v}, \quad \frac{\partial^2\theta_i}{\partial u\partial v}=c\theta_i, \\ &\frac{\partial^2\theta_i}{\partial v^2}=\frac{e_i-1}{e_i+1}\frac{\eta_i}{c}\frac{\partial\theta_i}{\partial u}+\frac{\partial \log c}{\partial v}\frac{\partial\theta_i}{\partial v}. \end{aligned} \quad (i=1,2) \right\} \quad (47)$$

Thus far we suppose ε_i and η_i capable of assuming either values 0 or 1.

By means of θ_1 and θ_2 we effect transformations of N into N_1 and N_2 respectively. From the form of (12) it follows that there exists functions θ_{12} and θ_{21} defined by

$$\frac{\partial}{\partial u}(\theta_i\theta_{ij})=-\theta_i\frac{\partial\theta_j}{\partial u}+\theta_j\frac{\partial\theta_i}{\partial u},\quad \frac{\partial}{\partial v}(\theta_i\theta_{ij})=\theta_i\frac{\partial\theta_j}{\partial v}-\theta_j\frac{\partial\theta_i}{\partial v}.\quad \left(\begin{matrix}i=1,2,\\ j=1,2,\end{matrix}\ i\neq j\right) \tag{48}$$

Evidently the constants of integration in these equations can be chosen so that

$$\theta_1\theta_{12}+\theta_2\theta_{21}=0.$$

Hereafter we assume that the functions θ_{12} and θ_{21} are paired in this way, and there are ∞^1 such pairs.

The functions θ_{12} and θ_{21} can be used to effect transformations T of N_1 and N_2 respectively into the same net N_{12}.* We seek the conditions that N_{12} shall be a net of period 3.

From the preceding investigation it follows that θ_{12} must satisfy the equations

$$\frac{\partial^2\theta_{12}}{\partial u^2}=\frac{\partial\log c_1}{\partial u}\frac{\partial\theta_{12}}{\partial u}+\frac{e_{12}+1}{e_{12}-1}\frac{\varepsilon_{12}}{c_1}\frac{\partial\theta_{12}}{\partial v},\quad \frac{\partial^2\theta_{12}}{\partial v^2}=\frac{e_{12}-1}{e_{12}+1}\frac{\eta_{12}}{c_1}\frac{\partial\theta_{12}}{\partial u}+\frac{\partial\log c_1}{\partial v}\frac{\partial\theta_{12}}{\partial v}, \tag{49}$$

where c_1 is given by (30), and ε_{12} and η_{12} are 0 or 1, as the case may be.

If equations (48) be differentiated and the expressions for $\dfrac{\partial^2\theta_{12}}{\partial u^2}$ and $\dfrac{\partial^2\theta_{12}}{\partial v^2}$ be substituted in (49), the resulting equations are reducible to

$$\left.\begin{aligned}
&\theta_{12}\left(\frac{e_1+1}{e_1-1}\varepsilon_1+\frac{e_{12}+1}{e_{12}-1}\varepsilon_{12}\right)+\frac{2\varepsilon_1}{c\theta_1}\frac{e_1+1}{e_1-1}\frac{\partial\theta_1}{\partial v}\frac{\partial\theta_2}{\partial u}-\frac{2\varepsilon_2}{c\theta_1}\frac{e_2+1}{e_2-1}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}\\
&+\theta_2\left(\frac{e_{12}+1}{e_{12}-1}\varepsilon_{12}-\frac{e_1+1}{e_1-1}\varepsilon_1\right)+\theta_1\left(\frac{e_2+1}{e_2-1}\varepsilon_2-\frac{e_{12}+1}{e_{12}-1}\varepsilon_{12}\right)\frac{\partial\theta_2}{\partial v}\Big/\frac{\partial\theta_1}{\partial v}=0,\\
&\theta_{12}\left(\frac{e_1-1}{e_1+1}\eta_1+\frac{e_{12}-1}{e_{12}+1}\eta_{12}\right)+\frac{2\eta_2}{c\theta_1}\frac{e_2-1}{e_2+1}\frac{\partial\theta_1}{\partial v}\frac{\partial\theta_2}{\partial u}-\frac{2\eta_1}{c\theta_1}\frac{e_1-1}{e_1+1}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}\\
&+\theta_2\left(\frac{e_1-1}{e_1+1}\eta_1-\frac{e_{12}-1}{e_{12}+1}\eta_{12}\right)+\theta_1\left(\frac{e_{12}-1}{e_{12}+1}\eta_{12}-\frac{e_2-1}{e_2+1}\eta_2\right)\frac{\partial\theta_2}{\partial u}\Big/\frac{\partial\theta_1}{\partial u}=0.
\end{aligned}\right\} \tag{50}$$

We consider first the case when N is of the first type. As we have seen in § 7, the nets N_1, N_2 and N_{12} are then of the first type. Thus all the ε's and η's are equal to 1. Making this substitution in (50), and eliminating θ_{12} from the resulting equations, we get

$$(e_{12}-e_2)\left(\frac{e_1+1}{e_2+1}\frac{\partial\theta_2}{\partial u}\frac{\partial\theta_1}{\partial v}-\frac{e_1-1}{e_2-1}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}\right)c_1=0.$$

* *Transactions, loc. cit.*, §§ 5, 11.

It is readily shown that the expression in the parenthesis vanishes only when $\theta_2/\theta_1=\text{const.}$, but in this case N_1 and N_2 are the same net. Since $c_1\neq0$, we have $e_{12}=e_2$. Then equations (50) reduce to the single one

$$(e_1e_2-1)\theta_1\theta_{12}+\frac{(e_1+1)(e_2-1)}{c}\frac{\partial\theta_1}{\partial v}\frac{\partial\theta_2}{\partial u}-\frac{(e_2+1)(e_1-1)}{c}\frac{\partial\theta_1}{\partial u}\frac{\partial\theta_2}{\partial v}$$
$$+\theta_1\theta_2(e_1-e_2)=0. \quad (51)$$

This value satisfies equations (48). Hence we have the theorem:

If N_1 and N_2 are nets of the first type, which are transforms of a net N of this type, there exists a unique net N_{12} of the first type, which is a transform of N_1 and N_2; and it can be found without quadratures.

From the equations of the theorem of permutability of transformations T * we find that for the present choice of coordinates the coordinates of N_{12} are given by

$$\theta_{12}(x_{12}-x)=\theta_2(x_2-x_1). \quad (52)$$

We suppose now that the nets N, N_1 and N_2 are of the second type. If N_{12} also is to be of this type, the ϵ's must have the value zero, and the η's one. Then the first of (50) is satisfied identically, and if the second be differentiated with respect to u, and use is made of (48), the result is reducible to

$$\frac{\partial\theta_1}{\partial u}\frac{e_2-e_{12}}{(e_2+1)(e_{12}+1)}=0.$$

Hence $e_{12}=e_2$ and the second of (50) reduces to (51). As this value satisfies the second of (48), we have the theorem:

If N_1 and N_2, nets of the second type, are transforms of a net N of the second type, there exists a unique net N_{12} of the second type which is their transform.

We consider finally the possibility of all four nets N, N_1, N_2, N_{12} being of the third type. In this case equations (50) are satisfied identically, and consequently all the nets N_{12} are of the third type. Let the functions θ_1 and θ_2 determining the transforms N_1 and N_2 be, according to (43),

$$\theta_i=[f_i(1-uv)+g_iu+h_iv](1+uv) \qquad (i=1,2).$$

By differentiation we have

$$\frac{\partial\theta_i}{\partial u}=\frac{-2f_iv+g_i-h_iv^2}{(1+uv)^2},\quad \frac{\partial\theta_i}{\partial v}=\frac{-2f_iu+h_i-g_iu^2}{(1+uv)^2}.$$

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* *Transactions*, p. 107.

Substituting these values in (48) and making use of the abbreviation $(ab) \equiv a_1b_2 - a_2b_1$, we get

$$\frac{\partial}{\partial u}(\theta_1\theta_{12}) = \frac{-(fg)-(fh)v^2+(gh)v}{(1+uv)^2}, \quad \frac{\partial}{\partial v}(\theta_1\theta_{12}) = \frac{(fh)+(fg)u^2+(gh)u}{(1+uv)^2}.$$

The integral of these equations is

$$\theta_1\theta_{12} = \frac{-\frac{1}{2}(gh)(1-uv)-(fg)u+(fh)v}{1+uv} + \text{const.}$$

Hence we have the theorem:

If N is a net of the third type, and N_1 and N_2 are two transforms of this type, there can be found without quadrature ∞^1 nets N_{12} which are transforms of N_1 and N_2.

12. **Surfaces of Tzitzeica.** When the homogeneous point coordinates of a net of period 3 are chosen so that they satisfy equations (24), they are the non-homogeneous coordinates of a surface referred to its asymptotic lines. In case $\epsilon=\eta=0$, the surface is a central quadric. When $\eta=1$, the surface is ruled or not according as $\epsilon=0$ or 1. These surfaces were discovered by Tzitzeica * in his search for surfaces whose total curvature at each point is in constant ratio with the fourth power of the distance of the tangent plane at the point from a point fixed in space, and constitute the complete solution of the problem.

We have seen in § 8 that the complete determination of the ruled surfaces of Tzitzeica requires the integration of an equation of the third order. Making use of this result and the expressions for the coordinates as there given, Tzitzeica showed † that these ruled surfaces are characterized by the property that their flexnode curve is at infinity.

Since the transformation of the planar nets as given by (33) is reciprocal in character, it follows that the equations of the inverse transformation are of the same form. Interpreted for the surface whose non-homogeneous coordinates are $x^{(1)}, x^{(2)}, x^{(3)}$, we have transformations of these surfaces into surfaces of the same kind, such that a surface and a transform are the focal surfaces of a W-congruence. Tzitzeica ‡ announced, without proof, the existence of these transformations.

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* *Comptes Rendus*, Vol. CXLIV (1907), p. 1257; Vol. CXLV (1907), p. 1132. Also *Rendiconti di Palermo*, Vol. XXV (1908), pp. 180–189.

† *Comptes Rendus*, Vol. CXLV, *loc. cit.*

‡ *Comptes Rendus*, Vol. CL (1910), pp. 955, 1227,

*Orthogonal Function Sets Arising from Integral Equations.**

By O. D. Kellogg.

1. *Introduction.*

As is well known,† $K(x, y)$ being a real, continuous, symmetric function, not identically zero, on the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, there is at least one value of λ for which the integral equation

$$\phi(x) = \lambda \int_0^1 \phi(y) K(x, y) dy \tag{1}$$

has a solution, $\phi(x)$, continuous, and not identically zero. Unless $K(x, y)$ is the sum of a finite number of products like $\pm\phi(x)\phi(y)$, there is an infinite number of such values of λ, and, in case it converges uniformly, the following development holds for $K(x, y)$:

$$K(x, y) = \frac{\phi_0(x)\phi_0(y)}{\lambda_0} + \frac{\phi_1(x)\phi_1(y)}{\lambda_1} + \frac{\phi_2(x)\phi_2(y)}{\lambda_2} + \ldots, \tag{2}$$

where $\phi_i(x)$ is the normed solution of (1) corresponding to λ_i. The functions $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, ..., form an orthogonal set on the interval $(0, 1)$. They will be called the "harmonics" of the kernel $K(x, y)$, and the corresponding values of λ_i, their "frequencies." The "iterated kernels" are defined by

$$K_j(x, y) = \int_0^1 K_{j-1}(x, r) K(r, y) dr, \text{ and } K_0(x, y) = K(x, y). \tag{3}$$

For these, the developments, known to be uniformly convergent ‡ for $i \geqq 2$, hold:

$$K_j(x, y) = \frac{\phi_0(x)\phi_0(y)}{\lambda_0^{j+i}} + \frac{\phi_1(x)\phi_1(y)}{\lambda_1^{j+i}} + \frac{\phi_2(x)\phi_2(y)}{\lambda_2^{j+i}} + \ldots. \tag{4}$$

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* Presented to the American Mathematical Society, September 4, 1917.

† See, for instance, Schmidt, "Entwickelung wilkürlicher Functionen nach Systemen vorgeschriebener," Diss. Göttingen, 1915; Bôcher, "An Introduction to the Study of Integral Equations," Cambridge tracts in math. and math. phys., 1909; Kowalewski, "Einführung in die Determinantentheorie," Veit u. Comp., Leipzig, 1909.

‡ Kowalewski, *loc. cit.*, p. 533.

In a previous paper* I have shown that a number of the oscillation properties of the more common orthogonal sets are consequences of continuity, orthogonality, and the added condition:

(D): $D_n(x_0, x_1, \ldots, x_n) > 0$ on the range R: $0 < x_0 < x_1 \ldots x_n < 1$, for $n = 0, 1, 2, \ldots$, where

$$D_n(x_0, x_1, \ldots, x_n) = \begin{vmatrix} \phi_0(x_0), \phi_1(x_0), \ldots, \phi_n(x_0), \\ \phi_0(x_1), \phi_1(x_1), \ldots, \phi_n(x_1), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \phi_0(x_n), \phi_1(x_n), \ldots, \phi_n(x_n), \end{vmatrix}$$

and $D_0(x_0) = \phi_0(x_0)$.

It was stated that it appeared desirable to connect this condition with the theory of integral equations. The following is intended as a contribution to this point.

2. *The Condition on $K(x, y)$.*

To find the necessary and sufficient condition on $K(x, y)$ in order that its harmonics may have the property (D) presents difficulties, since the same harmonics may arise from a variety of kernels, the frequencies being altered. However, a condition may be found which is simple, and which appears to be satisfied in the more common cases. We proceed to indicate considerations which suggest this condition, and which make it appear useful.

It is a property of the more common sets that if $\Phi_{0n}(x) = c_0\phi_0(x) + c_1\phi_1(x) + \ldots + c_n\phi_n(x)$ is a linear combination of the harmonics of $K(x, y)$, the function $\int_0^1 \Phi_{0n}(y) K(x, y)\,dy$ has no more sign changes in the interior of $(0, 1)$ than $\Phi_{0n}(x)$, and in many cases, this property holds for any continuous function $f(x)$. Assuming that it does, let $y_0, y_1, \ldots, y_n$ be any $n+1$ points on R. Choose for $f(x)$ any continuous function $f(x, \varepsilon)$ equal to 0 except on the sub-intervals $(y_i - \varepsilon, y_i + \varepsilon)$, where it is determined so that

$$\int_{y_i - \epsilon}^{y_i + \epsilon} f(x, \varepsilon)\,dx = (-1)^{i-1} c_i,$$

the c_i being independent of ε, and $f(x, \varepsilon)$ not changing sign on any sub-interval. If the c_i are all of one sign, $f(x, \varepsilon)$ will have n sign changes, otherwise fewer.

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* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXVIII, No. 1 (1916), p. 1. The condition (D) employed in that paper, and in this, which requires determinants of the functions $\phi_i(x)$ to be positive, is in reality no more restrictive than that they be different from zero, as is evident when we reflect that the function $\phi_n(x)$ may be replaced, if necessary, by its negative, without affecting any other hypothesis. But it is more convenient to retain it in the original form.

If now the law of the mean be applied to the integral $\int_0^1 f(y, \varepsilon)K(x, y)dy$, and the limit taken as $\varepsilon=0$, the result is

$$f_1(x)=c_0K(x, y_0)-c_1K(x, y_1)+\ldots.+(-1)^{n-1}c_nK(x, y_n),$$

and it may be shown, on our assumption, that this has no more sign changes that $f(x, \varepsilon)$. In other words, there is no set of positive numbers $a_0, a_1, \ldots.,$ a_{n+1}, and no set of points $x_0, x_1, \ldots., x_{n+1}$ on R for which the following equations are possible:

$$\left.\begin{array}{r} c_0K(x_0, y_0)-c_1K(x_0, y_1)+\ldots.+(-1)^nc_nK(x_0, y_n)=a_0, \\ -[c_0K(x_1, y_0)-c_1K(x_1, y_1)+\ldots.+(-1)^nc_nK(x_1, y_n)=a_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ (-1)^{n+1}[c_0K(x_{n+1}, y_0)-c_1K(x_{n+1}, y_1)+\ldots.+(-1)^nc_nK(x_{n+1}, y_n)=a_{n+1}. \end{array}\right\} \quad (5)$$

Now let $M_0, M_1, \ldots., M_{n+1}$ be the determinants formed from the matrix

$$\|K(x_i, y_j)\|, \quad i=0, 1, \ldots., n+1, \quad j=0, 1, \ldots., n \qquad (6)$$

by omitting the first, second, third rows, and so forth. No two of the quantities M_i can differ in sign, since in the contrary case the equation $M_0a_0+M_1a_1 +\ldots.+M_{n+1}a_{n+1}=0$ would be possible with all a_i positive, and for such a_i the equations (5) would be compatible. So we conclude that no two of the determinants

$$K\begin{pmatrix}x_0, x_1, \ldots., x_n\\ y_0, y_1, \ldots., y_n\end{pmatrix}=\begin{vmatrix} K(x_0, y_0), & K(x_0, y_1), & \ldots., & K(x_0, y_n) \\ K(x_1, y_0), & K(x_1, y_1), & \ldots., & K(x_1, y_n) \\ \ldots & \ldots & \ldots & \ldots \\ K(x_n, y_0), & K(x_n, y_1), & \ldots., & K(x_n, y_n) \end{vmatrix} \qquad (7)$$

taken from the same matrix (6), can have opposite signs. If, now, we can pass through a chain of such determinants, different from 0, from any given one to any given second, we may conclude that the determinant (7) is either never negative, or never positive on the region RR: $0<x_0<x_1\ldots.<x_n<1$. $0<y_0<y_1\ldots.<y_n<1$. Thus is indicated the condition of the next paragraph,

3. *A Sufficient Condition Upon* $K(x, y)$.

We shall suppose the function $K(x, y)$, in addition to being real, continuous, and symmetric on the square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, to have the property:

(K) : $K\begin{pmatrix}x_0, x_1, \ldots., x_n\\ y_0, y_1, \ldots., y_n\end{pmatrix}\geqq 0$ on RR for $n=0, 1, 2, \ldots.$, the *inequality* holding when $x_i=y_i$ for all i. By $K\begin{pmatrix}x_0\\ y_0\end{pmatrix}$ is understood $K(x_0, y_0)$.

If this property is not enjoyed by $K(x, y)$, it may be by one of the iterated kernels (3), which has the same harmonics. We proceed to prove the theorem:

If $K(x, y)$ is real, continuous, and symmetric on the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, and has the property (K), its harmonics have the property (D), with, of course, all the properties derived in the paper referred to.

We begin with the following special case as a lemma:

If $K(x, y)$ is real, continuous and symmetric and satisfies (K) for $n=0$, it has but one frequency of least absolute value, and but one harmonic corresponding to this frequency. This harmonic never vanishes on the interior of the interval $(0, 1)$, and its frequency is positive.

It should be noticed that the property (K) for $n=0$ is retained in the iterated kernels. We start from the first iterated kernel, $K_1(x, y)$, whose frequencies, being the squares of those of $K(x, y)$ are all positive. Its least frequency is thus λ_0^2, which we may assume to be 1 without impairing the argument, since we may replace $K_1(x, y)$ by $\lambda_0^2 K_1(x, y)$, the latter kernel having 1 as least frequency. If more than one harmonic corresponded to 1, the development (2) would take the form:

$$K_1(x, y) = \phi_0(x)\phi_0(y) + \phi_1(x)\phi_1(y) + \dots + \phi_n(x)\phi_n(y)$$
$$+ \frac{\phi_{n+1}(x)\phi_{n+1}(y)}{\lambda_{n+1}^2} + \frac{\phi_{n+2}(x)\phi_{n+2}(y)}{\lambda_{n+2}^2} + \dots, \quad (8)$$

where in the first line appear the harmonics corresponding to 1, and in the second, those corresponding to frequencies greater than 1. Repeated iteration diminishes these later terms, and in the limit we have a function

$$F(x, y) = \phi_0(x)\phi_0(y) + \phi_1(x)\phi_1(y) + \dots + \phi_n(x)\phi_n(y), \quad (9)$$

which is continuous, never negative, and symmetric. The harmonics composing it may be assumed to be normed and orthogonal. $F(x, y)$ may, moreover, be shown to be *positive throughout the whole interior of the square* $0 < x < 1$, $0 < y < 1$, as follows. It has positive elements, since it is not identically 0, and it is, for any fixed y, a solution of the integral equation $f(x) = \int_0^1 f(y) K_1(x, y)\, dy$. Let b be a value of y such that $f(x) = F(x, b)$ is positive at some point $0 < x < 1$. Then $f(x)$ can not be 0 at any interior point, for if it were, there would be an interior point, a, terminating an interval on which $f(x) > 0$, and this would lead to the contradiction

$$f(a) = 0 = \int_0^1 f(y) K_1(a, y)\, dy,$$

the integrand being positive near $y=a$, and never negative. Thus $F(x, y)>0$ at all points of the line $y=b$, and being symmetric, at all points of $x=b$. Hence, the argument being repeated with any other value of y, the conclusion is established. We shall now show that $F(x, y)$ being >0 within the square, the sum (9) reduces to a single product.

We must distinguish three cases according to the behavior of $F(x, y)$ on the sides of the square. We note that the equation holds:

$$F(x, y)=\int_0^1 F(x, r) F(r, y) dr. \tag{10}$$

If $F(x, y)$ vanishes at a boundary point, say $(0, b)$, we have

$$0=\int_0^1 F(0, r) F(r, b) dr,$$

and as the integrand is never negative, it must vanish for all r. If $0<b<1$, $F(r, b)>0$ for all interior r, and $F(0, r)=0$. If $b=0$, the integrand is a square, and the same conclusion follows. If $b=1$, either one of the factors must vanish on the whole open interval, or one factor must vanish at some points, and the other at all the rest at least. But the last alternative is impossible, since (10) will show that if $F(x, y)$ is positive at one boundary point, it will be positive on the whole open side. We conclude that if $F(x, y)$ vanishes at any boundary point between corners, it vanishes on the whole side corresponding, and by symmetry, on a second side. The cases, then, are: I, $F(x, y)>0$ on the closed square; II, $F(x, y)>0$ on the open square, but vanishing on the whole boundary; and III, $F(x, y)>0$ on the open square and on two symmetric sides, but vanishing on the other two.

If b is any interior point, $F(x, b)$ is a harmonic, since (9) shows it to be a linear homogeneous combination of harmonics with constant coefficients. It may then be normalized and considered one of the harmonics $\phi_i(x)$, since if the functions of (9) be subjected to an orthogonal transformation, its form remains unaltered, the harmonics remaining normed and orthogonal. We may therefore identify $\phi_0(x)$ with $F(x, b)/\sqrt{\int_0^1 F^2(x, b) dx}$.

CASE I. Here $\phi_0(x)>0$ on the closed interval $(0, 1)$. If there is a second function $\phi_1(x)$, its ratio to $\phi_0(x)$ is continuous on the closed interval, and hence attains its maximum, M, say for $x=a$. Then $M\phi_0(x)-\phi_1(x)\geqq 0$ on $(0, 1)$. But as

$$M\phi_0(x)-\phi_1(x)=\int_0^1 F(x, y)\left[M\phi_0(y)-\phi_1(y)\right] dy, \tag{11}$$

we have, for $x=a$, a contradiction, since the integrand is continuous, has positive elements, and is never negative, while the left-hand side vanishes.

CASE II. Here, $F(x, y)$ being uniformly continuous, and vanishing on the whole boundary, it is possible, given $\varepsilon>0$, to find $\delta>0$, such that $0\leqq F(x, y)<\delta$ for $0\leqq x\leqq\delta$, and all y, and for $1-\delta\leqq x\leqq 1$ and all y, and for the corresponding regions formed by interchanging x and y. A similar condition holds for all harmonics as a consequence. For, for $0\leqq x\leqq\delta$, or $1-\delta\leqq x\leqq 1$, $\phi_i(x) =\int_0^1 F(x, y)\phi_i(y)dy$ is less in absolute value than

$$\varepsilon\int_0^1|\phi_i(y)|\,dy\leqq\varepsilon\sqrt{\int_0^1\phi_i^2(y)\,dy}=\varepsilon.$$

Now, if there be a second harmonic, the ratio $\phi_1(x)/\phi_0(x)$ is continuous on the closed interval $(\delta, 1-\delta)$, and attains its maximum M, say for $x=a$ within this interval. Then $M\phi_0(x)-\phi_1(x)\geqq 0$ on $(\delta, 1-\delta)$, and on the rest of $(0, 1)$, since $\phi_0(x)\geqq 0$, $M\phi_0(x)-\phi_1(x)>-\varepsilon$. But $M\phi_0(x)-\phi_1(x) =\int_0^1 F(x, y)\,[M\phi_0(y)-\phi_1(y)]\,dy=\int_\delta^{1-\delta}+\int_0^\delta+\int_{1-\delta}^1$, and as the first integral is never negative, we conclude that for $0\leqq x\leqq\delta$, or $1-\delta\leqq x\leqq 1$, $M\phi_0(x) -\phi_1(x)>-2\delta\varepsilon^2$. If the process is repeated, we see that this function is greater than $-(2\delta\varepsilon)^2\varepsilon$, and so on, so that the lower limit of $M\phi_0(x)-\phi_1(x)$ can only be 0 on the whole interval $(0, 1)$. We may then complete the reasoning as in Case I.

CASE III. A combination of the methods employed in the first two cases leads to the same result, that there can not be a second harmonic.

Thus $\phi_0(x)$, except for a multiplicative constant, is the only harmonic of $K_1(x, y)$ of frequency 1, and therefore the only harmonic of $K(x, y)$ of frequency $+1$ or -1. We have seen that it does not vanish within $(0, 1)$. Since it satisfies the integral equation (1) its frequency must be $+1$, because the integral and the left member of the equation have the same signs. This completes the proof of the lemma.

It is important for what follows to notice that though the lemma is proven for a kernel in two variables, the proof can be easily adapted to a real, continuous kernel symmetric in two sets of variables.

The lemma has established the property (D) for the harmonics $\phi_i(x)$ for $n=0$. To extend the argument to all values of n, we proceed to derive integral equations which the determinants D_1, D_2, ... satisfy, and apply the lemma to their kernels.

An abbreviation will be helpful. Let ξ stand for the aggregate of variables $x_0, x_1, \ldots, x_n$; η for $y_0, y_1, \ldots, y_n$; and let us write $\varkappa(\xi, \eta)$ for $K\begin{pmatrix} x_0, x_1, \ldots, x_n \\ y_0, y_1, \ldots, y_n \end{pmatrix}$; and $\Delta(\xi)$ for $D_n(x_0, x_1, \ldots, x_n)$. Let S stand for the region $0 \leqq x_i \leqq 1$, $i=0, 1, \ldots, n$, and by $\int_S f(\xi)\,d\xi$ let us understand

$$\int_0^1 \int_0^1 \cdots \int_0^1 f(x_0, x_1, \ldots, x_n)\,dx_0, dx_1, \ldots, dx_n.$$

4. *The Integral Equation for* $\Delta(\xi)$.

We apply to $\Delta(\eta)\varkappa(\xi, \eta)$ Lagrange's product formula:

$$\Delta(\xi)\varkappa(\xi, \eta) = \begin{vmatrix} \sum_0^n \phi_0(y_i)K(x_0, y_i), & \sum_0^n \phi_0(y_i)K(x_1, y_i), & \ldots, & \sum_0^n \phi_0(y_i)K(x_n, y_i) \\ \sum_0^n \phi_1(y_i)K(x_0, y_i), & \sum_0^n \phi_1(y_i)K(x_1, y_i), & \ldots, & \sum_0^n \phi_1(y_i)K(x_n, y_i) \\ \cdots & \cdots & \cdots & \cdots \\ \sum_0^n \phi_n(y_i)K(x_0, y_i), & \sum_0^n \phi_n(y_i)K(x_1, y_i), & \ldots, & \sum_0^n \phi_n(y_i)K(x_n, y_i) \end{vmatrix}$$

$$= \Sigma \begin{vmatrix} \phi_0(y_{i_0})K(x_0, y_{i_0}), & \phi_0(y_{i_1})K(x_1, y_{i_1}), & \ldots, & \phi_0(y_{i_n})K(x_n, y_{i_n}) \\ \phi_1(y_{i_0})K(x_0, y_{i_0}), & \phi_1(y_{i_1})K(x_1, y_{i_1}), & \ldots, & \phi_1(y_{i_n})K(x_n, y_{i_n}) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_n(y_{i_0})K(x_0, y_{i_0}), & \phi_n(y_{i_1})K(x_1, y_{i_1}), & \ldots, & \phi_n(y_{i_n})K(x_n, y_{i_n}) \end{vmatrix},$$

where in the summation $i_0, i_1, \ldots, i_n$ take on all values from 0 to n. However, if any two of the indices are equal, the corresponding determinant is 0, two columns becoming proportional, so that the indices may be restricted to the $(n+1)!$ permutations of the numbers $0, 1, \ldots, n$. Thus we have $(n+1)!$ terms, which, after integration over the field S, all become equal, namely to

$$\begin{vmatrix} \phi_0(x_0)/\lambda_0, & \phi_0(x_1)/\lambda_0, & \ldots, & \phi_0(x_n)/\lambda_0, \\ \phi_1(x_0)/\lambda_1, & \phi_1(x_1)/\lambda_1, & \ldots, & \phi_1(x_n)/\lambda_1, \\ \cdots & \cdots & \cdots & \cdots, \\ \phi_n(x_0)/\lambda_n, & \phi_n(x_1)/\lambda_n, & \ldots, & \phi_n(x_n)/\lambda_n. \end{vmatrix}$$

Hence the required integral equation for $\Delta(\xi)$ is

$$\chi(\xi) = \mu \int_0^1 \chi(\eta)\varkappa(\xi, \eta)\,d\eta, \tag{12}$$

the solution $\Delta(\xi)$ corresponding to $\mu = (\lambda_0\lambda_1 \ldots \lambda_n)/(n+1)!$.

Various other solutions are obtained by forming determinants of sets of $n+1$ of the functions $\phi_i(x)$, other than the first $n+1$, the frequencies being the products of the corresponding frequencies of the $\phi_i(x)$ divided by $(n+1)!$

In order to apply the lemma of the last paragraph, it is necessary to know that the frequency of $\Delta(\xi)$ is the one of smallest absolute value of the kernel $\varkappa(\xi,\eta)$. To do this, we shall show that $\varkappa(\xi,\eta)$ has no other frequencies than those of the form $=\pm(\lambda_{i_0}\lambda_{i_1}\ldots\lambda_{i_n})/(n+1)!$. This is the subject of the next paragraph.

5. *The Frequencies of* $\varkappa(\xi,\eta)$.

We shall here make use of the following fact: $\lambda_0, \lambda_1, \ldots,$ *being a set of real quantities, they are determined by the power sums* $s_{-1}=\Sigma 1/\lambda_i$, $s_{-2}=\Sigma 1/\lambda_i^2, \ldots,$ *in case these sums are absolutely convergent.* The truth of this fact follows from our knowledge that the power sums determine a transcendental integral function whose roots the λ_i are.* Let us now assume, the justification to follow later, that $K(x,y)$ is the first iterated kernel of a real continuous symmetric kernel. It will follow from (4) that the power sums all converge absolutely, and will be given by

$$s_{-j}=\int_0^1 K_{j-1}(r,r)\,dr=\sum_i 1/\lambda_i^j. \tag{13}$$

Under the assumption just made, $\varkappa(\xi,\eta)$ will also be the first iterated kernel of a real, symmetric, continuous kernel and the power sums of its frequencies, also absolutely convergent, will be given by

$$s'_{-j}=\int_s \varkappa_{j-1}(\rho,\rho)\,d\rho=\sum_i 1/\mu_i^j. \tag{14}$$

We are concerned with establishing the equations:

$$s'_{-j}=\sum_{i_0,i_1,\ldots,i_n}\left[\frac{(n+1)!}{\lambda_{i_0}\lambda_{i_1}\ldots\lambda_{i_n}}\right]^j, \qquad j=1,2,\ldots\ldots \tag{15}$$

In order to prove this formula for $j=1$, we introduce the notation p_n, defining it by the equation:

$$(n+1)!\,p_{n+1}=\int_S K\begin{pmatrix} r_0, r_1, \ldots, r_n \\ r_0, r_1, \ldots, r_n\end{pmatrix} dr_0dr_1\ldots dr_n$$

$$=\int_0^1\int_0^1\ldots\int_0^1 \begin{vmatrix} K(r_0,r_0), K(r_0,r_1), \ldots, K(r_0,r_n) \\ K(r_1,r_0), K(r_1,r_1), \ldots, K(r_1,r_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ K(r_n,r_0), K(r_n,r_1), \ldots, K(r_n,r_n)\end{vmatrix} dr_0dr_1\ldots dr_n, \tag{16}$$

and shall prove that p_n is the sum of the products of the reciprocals of the λ_i n at a time. It will be noticed that the terms of the expanded determinant

* Log $P=\log(1-z/\lambda_0)(1-z/\lambda_1)(1-z/\lambda_2)\ldots$ has these sums, apart from numerical factors, as coefficients, when expanded in a power series.

contain only products of the form $K(r_{i_1}, r_{i_2})K(r_{i_2}, r_{i_3})\ldots.K(r_{i_{k-1}}, r_{i_k})K(r_{i_k}, r_{i_1})$, which, upon integration becomes $\int_0^1 K_{k-1}(r, r)\,dr, = s_{-k}$. Fixing our attention upon those products containing a factor from the first row and one from the first column, we easily verify the following recursion formula for p_{n+1}:

$$(n+1)\,p_{n+1} = s_{-1}p_n - s_{-2}p_{n-1} + s_{-3}p_{n-2} - \ldots. + (-1)^n s_{-1}.$$

But this is the recursion formula for the product sums in terms of the power sums given in the theory of algebraic equations, and applicable here because of the convergence of the power sums. And as (16) also gives $p_1 = s_{-1}$, it is clear that p_n is the product sum stated.*

On the other hand, $(n+1)\,!\,p_{n+1}$, by (16) is $\int_S \varkappa(\rho, \rho)\,d\rho = s'_{-1}$, so that (15) is proved for $j=1$.

To extend the proof is not difficult. We compute the value of the integral $\varkappa_1(\xi, \eta) = \int_S \varkappa(\xi, \rho)\varkappa(\rho, \eta)\,d\eta$, just as we did $\int_S \Delta(\eta)\varkappa(\xi, \eta)\,d\eta$ in the last paragraph, obtaining

$$\varkappa_1(\xi, \eta) = (n+1)\,! \begin{vmatrix} K_1(x_0, y_0), & K_1(x_0, y_1), & \ldots., & K_1(x_0, y_n) \\ K_1(x_1, y_0), & K_1(x_1, y_1), & \ldots., & K_1(x_1, y_n) \\ \ldots & \ldots & \ldots & \ldots \\ K_1(x_n, y_0), & K_1(x_n, y_1), & \ldots., & K_1(x_n, y_n) \end{vmatrix}, \qquad (17)$$

This shows that the power sum s'_{-2} may be computed just as was s'_{-1}, only replacing $K(x, y)$ by $K_1(x, y)$ and multiplying by $(n+1)\,!$. But the effect of replacing $K(x, y)$ by $K_1(x, y)$ is to replace the numbers λ_i by their squares. Thus the formula (15) is demonstrated for $j=2$. It is clear that the method is general, and we may regard (15) as true for all positive integers j.

Reverting to our assumption that $K(x, y)$ was the first iterated kernel of a continuous symmetric kernel, we note that should there be doubt of this, we may, in the above reasoning, begin with $K_1(x, y)$ instead of $K(x, y)$. Equation (17) then shows that $\varkappa(\xi, \eta)$ would be replaced by $\varkappa_1(\xi, \eta)/(n+1)\,!$, and instead of the frequencies of $\varkappa(\xi, \eta)$ being identified with the products $(\lambda_{i_0}\lambda_{i_1}\ldots.\lambda_{i_n})/(n+1)\,!$, we should only have their squares identified with the squares of these products. But this is sufficient for the purposes in hand, and gives the result announced at the close of the last paragraph.

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* The result has an interesting bearing on the Fredholm series for symmetric kernels, for p_n is the coefficient of $(-\lambda)^n$ in the series $d(\lambda) = 1 - \frac{\lambda}{1!}\int_0^1 K(r, r)\,dr + \frac{\lambda^2}{2!}\int_0^1\int_0^1 K\begin{pmatrix} r_0, & r_1 \\ r_0, & r_1 \end{pmatrix} dr_0\,dr_1 - \ldots.$, whose roots are the frequencies of $K(x, y)$. It shows that if the power sums are absolutely convergent, as they will be, for example, if $K(x, y)$ is the first iterated kernel of a real, symmetric, continuous kernel, the transcendental integral function $d(\lambda)$ *contains as factor no transcendental integral function without zeros.*

6. *The Property* (D).

Evidently the sign of $\Delta(\xi)$ is not constant on S, as an interchange of two arguments changes the sign of $\Delta(\xi)$. But we are concerned only with showing it different from 0 on R. We therefore substitute in the equation (12), the field R for the field S. In doing so, we note that $\Delta(\eta)$ and $\varkappa(\xi,\eta)$ are alternating functions on S for η, and that therefore their product is symmetric. Hence the field R is one of $(n+1)!$ symmetric sub-fields for the integral, corresponding to one of the $(n+1)!$ orders of the arguments y. Hence $\int_S \Delta(\eta)\varkappa(\xi,\eta)d\eta=(n+1)!\int_R \Delta(\eta)\varkappa(\xi,\eta)d\eta$, so that $\Delta(\xi)$ now satisfies the equation

$$\chi(\xi)=\nu\int_R \chi(\eta)\varkappa(\xi,\eta)d\eta \tag{18}$$

for $\nu=\lambda_0\lambda_1\ldots\lambda_n$. This equation will have the same harmonics as (12), and its frequencies will evidently be those of (12) multiplied by $(n+1)!$.

The hypothesis (K) on $K(x,y)$ includes the hypothesis (K) for $n=0$ on $\varkappa(\xi,\eta)$, and the lemma of paragraph 3 applies. As the frequency of least absolute value of $\varkappa(\xi,\eta)$ is $\pm\lambda_0\lambda_1\ldots\lambda_n$, since this is the smallest (or one of the smallest, in case several are equal) product of $n+1$ of the λ_i in absolute value, the single harmonic belonging to this frequency must be $\Delta(\xi)$. And this one harmonic does not vanish on R. Hence, by proper choice of the sign of $\phi_n(x)$, we may conclude $\Delta(\xi)=D_n(x_0,x_1,\ldots,x_n)>0$ on R, and the property (D) is thus generally established for the harmonics of $K(x,y)$.

7. *A Generalization.*

As, in the foregoing, we have used the hypothesis (K) only for one value of n, the following theorem follows:

If the real, continuous, symmetric kernel $K(x,y)$ *satisfies the hypothesis* (K) *for* $n=n_1, n_2, \ldots$, *then the harmonics of* $K(x,y)$ *satisfy the condition* (D) *for* $n=n_1, n_2, \ldots$.

In the next number of this Journal I shall establish the property (D) for the orthogonal function sets arising from ordinary linear homogeneous differential equations of the second order.

COLUMBIA, MO., *October* 20, 1917.

Complete Systems of Concomitants of the Three-Point and the Four-Point in Elementary Geometry.

By Charles Henry Rawlins, Jr.

Introduction.

In this discussion, two point-sets, containing three and four points respectively, are subjected to three transformations of elementary geometry; and complete systems of invariants and covariants, corresponding to the respective transformations, are derived. In the process we obtain some interesting geometric applications of the theory of binary forms and of symmetric functions.

PART I.

Invariants Under Translation.

Section (a): *The Three-Point.*

Consider first the three-point under translation. Let the points have complex coordinates α, β, γ, respectively, the roots of a cubic

$$a_0x^3+3a_1x^2+3a_2x+a_3=0.$$

Since translation is effected by adding to each root the same vector p, the invariants of the three-point under translation are the geometric equivalents of the invariants of the cubic under the transformation

$$x'=x+p.$$

Invariants under such a transformation are known, in the theory of binary forms, as *seminvariants*. Their complete system for the cubic is known* to consist of the coefficients of the cubic when so transformed that its second term vanishes, and the discriminant

$$\Delta\equiv(a_0a_3-a_1a_2)^2-4(a_0a_2-a_1^2)(a_1a_3-a_2^2).$$

Substituting $x-(a_1/a_0)$ for x in the cubic, we obtain

$$A_0x^3+3A_2x+A_3=0,$$

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* Elliott, "Algebra of Quantics," pp. 140, 162.

the form desired, where, except for factor a_0,

$$A_0=a_0, \quad A_2=a_0a_2-a_1^2, \quad A_3=a_0^2a_3-3a_0a_1a_2+2a_1^3.$$

Connecting these is the syzygy

$$A_0^2\Delta=4A_2^3+A_3^2$$

However, without loss of generality, we can consider A_0 (or a_0) as having the fixed value unity, in which case Δ is expressed *integrally* in terms of A_2 and A_3. Therefore

The complete system of invariants of the three-point under translation consists of A_2 and A_3.

Since the *shape* of the three-point is an invariant property under translation, the vanishing of A_2 or A_3 indicates some condition on the shape.

Vanishing of A_2:—If $A_2=0$, the cubic is

$$x^3+A_3=0,$$

with roots in ratio $1:\omega:\omega^2$, where ω is one of the complex cube roots of unity. The points form, therefore, the vertices of an equilateral triangle.

Vanishing of A_3:—If $A_3=0$, the cubic is

$$x^3+3A_2x=0$$

with roots in ratio $0:1:-1$. Hence the points are collinear, with one midway between the others.

A_1 being absent, the sum of the roots is zero and therefore the origin is at the centroid.

Section (b): The Four-Point.

Similarly, *the complete system of seminvariants of the four-point*

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0,$$

consists of *

$$A_0=a_0(=1), \quad A_2=a_0a_2-a_1^2, \quad A_3=a_0^2a_3-3a_0a_1a_2+2a_1^3,$$
$$I=a_0a_4-4a_1a_3+3a_2^2, \quad J=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3.$$

However, since

$$A_4=a_0^3a_4-4a_0^2a_1a_3+6a_0a_1^2a_2-3a_1^4=a_0^2I-3A_2^2,$$

we will use it instead of I.

Again the origin is at the centroid, and the vanishing of an invariant is a condition on the *shape* of the configuration.

Vanishing of A_2:—Let $\alpha, \beta, \gamma, \delta$ be the roots of

$$A_0x^4+6A_2x^2+4A_3x+A_4=0.$$

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* Elliott, *supra*, p. 170.

Then, if $A_2=0$, $\Sigma\alpha\beta=0$; so that, since $\overset{4}{\Sigma}\alpha=0$, then $\overset{4}{\Sigma}\alpha^2=0$. Refer the system to rectangular axes through the origin, so that $\alpha=X_1+iY_1$, $\beta=X_2+iY_2$, etc. Then

$$\sum_{j=1}^{4}(X_j+iY_j)^2=0.$$

Expanding and equating real parts,

$$\Sigma X^2-\Sigma Y^2=0.$$

But

$$\Sigma X^2+\Sigma Y^2=\Sigma R^2,$$

where the R's are distances from the origin to the respective points, constants in the present discussion. Therefore

$$\Sigma X^2=1/2\Sigma R^2=\text{constant}. \tag{1}$$

If the points are of unit mass, ΣX is the moment of inertia of the system about the Y-axis. But, the direction of this axis having been arbitrarily chosen, (1) tells us that the moment of inertia is the same for all axes through the centroid, thus making the "ellipse of inertia"* a circle.

Vanishing of A_3:—If $A_3=0$, the quartic is

$$x^4+6A_2x^2+A_4=0,$$

with factors x^2-m_1, x^2-m_2, and roots $\pm\sqrt{m_1}$, $\pm\sqrt{m_2}$. Hence the points form the vertices of a parallelogram.

Vanishing of A_4:—If $A_4=0$, one root of the quartic is zero. Hence one point lies at the centroid of the other three.

Vanishing of J:—The vanishing of J is known to be the condition that the four roots form harmonic pairs; or, geometrically, that the four points lie on one of a set of three mutually orthogonal circles, at its intersections with the other two.

Since *infinity* is unaltered by a finite translation, it has not been necessary, in the foregoing, to consider it as part of the apparatus employed.

PART II.

Monogenic Concomitants in the Complex Plane.

Section (a): General Theory.

We will consider next the monogenic concomitants of the point-sets in the complex plane, that is, concomitants not involving the conjugates of any of

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* Routh, "Rigid Dynamics," Chap. I.

the complex quantities used. Then the most general (linear) transformation possible is

$$x' = (ax+b)/(cx+d)$$

the product of an even number of inversions. Since infinity is not, in general, a fixed point of this transformation, it must be considered explicitly, that is:

We must discuss not merely the concomitants of a point-set, *but the concomitants of the point at infinity and the point-set.* Hence we use the theory of a binary cubic (quartic) and a linear form. From this theory we learn * that *the complete system of the cubic (quartic) and the linear form consists of the linear form, the complete system of the cubic (quartic) and the polars of this system with respect to the linear form.*

The linear form being the equation of infinity, its polar operator, under the present notation, is simply d/dx.

Section (b): The Three-Point.

The complete system of the cubic consists of †

the cubic itself: $C = a_0x^3 + 3a_1x^2 + 3a_2x + a_3$,

the Jacobian: $G = A_3x^3 + \ldots$,

the Hessian: $H = A_2x^2 + \ldots$,

the discriminant: $\Delta = (a_0a_3 - a_1a_2)^2 - 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2)$.

Successive derivatives of these are (neglecting constant factors):

$$C' = a_0x^2 + 2a_1x + a_2, \quad C'' = a_0x + a_1, \quad C''' = a_0 \text{ (negligible)},$$
$$G' = A_3x^2 + \ldots, \quad G'' = A_3x + \ldots, \quad G''' = A_3,$$
$$H' = A_2x + \ldots, \quad H'' = A_2.$$

As in Part I, Section (a), Δ is expressible integrally in terms of A_2 and A_3.

Complete System.—Thus the complete system contains

two cubics: C, G,

three quadratics: C', G', H,

three linear forms: C'', G'', H',

two invariants: $G''' = A_3$, $H'' = A_2$.

Geometric Equivalents.—We will state, without demonstration, the geometric equivalents (see figure).

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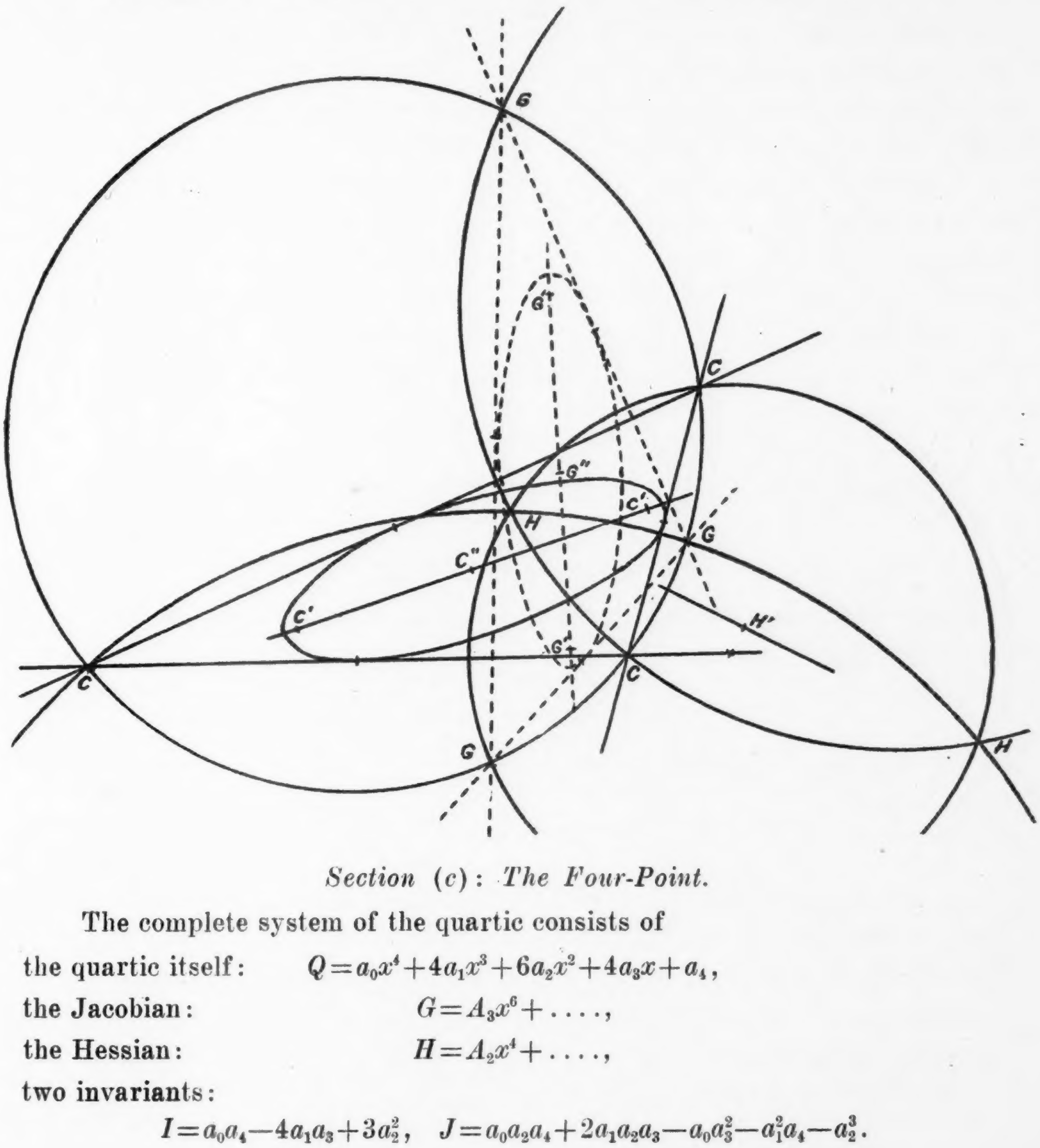
* Grace & Young, "Algebra of Invariants," 138 A.

† Salmon, "Higher Algebra"; Elliott and Grace & Young, *supra*.

The G-points are cut out of the circle on the C-points by means of the three Apollonian circles, that is, circles each on one C-point and ***drawn about*** the other two. The Apollonian circles are members of a pencil whose fixed points are the H-points. C' are the foci of the ellipse inscribed in the C-triangle with its center at the centroid, and C'' is the centroid. G' and G'' are similarly related to the G-triangle. H' is midway between the H-points. If $A_2=0$, an H-point is at infinity. If $A_3=0$, a G-point is at infinity.



Section (c): The Four-Point.

The complete system of the quartic consists of

the quartic itself: $Q=a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4$,

the Jacobian: $G=A_3x^6+\ldots\ldots$,

the Hessian: $H=A_2x^4+\ldots\ldots$,

two invariants:

$$I=a_0a_4-4a_1a_3+3a_2^2, \quad J=a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3.$$

Complete System.—Taking derivatives, the complete system of the four-point is

one sextic:	G,
one quintic:	G',
three quartics:	Q, G'', H,
three cubics:	Q', G''', H',
three quadratics:	Q'', $G^{(4)}$, H'',
three linear forms:	Q''', $G^{(5)}$, H''',
four invariants:	$G^{(6)}=A_3$, $H^{(4)}=A_2$, I, J.

Geometric Equivalents.—We will define a few of the geometric equivalents of these forms. G is the double points of the involutions formed by taking the Q-points in pairs. The derived forms can be interpreted by the following theorem:*

Given $\phi(p_1, p_2, \ldots, p_r)=0$, *homogeneous in the p's, as the equation of a curve, where the p's are the distances from given points* $a_1, a_2, \ldots, a_r$, *respectively, to a line of the curve, the foci of the curve are the roots of* $\phi(x-a_1, x-a_2, \ldots, x-a_r)=0$.

Suppose now n points, $a_1, a_2, \ldots, a_n$, given by

$$f=(x-a_1)(x-a_2)\ldots(x-a_n)=0.$$

The derived equation can be put in the form

$$f'=\frac{1}{x-a_1}+\frac{1}{x-a_2}+\ldots+\frac{1}{x-a_n}=0.$$

By the theorem, f' gives the foci of the curve

$$\sum_{i=1}^{n} 1/p_i=0,$$

which is of class $n-1$, is on the $n(n-1)/2$ joins of the n points and touches each join at its mid-point.

Hence G' is the foci of such a curve on the joins of the G-points, and similarly for the other derived forms.

If $A_2=0$, an H-point is at infinity. If $A_3=0$, a G-point is at infinity.

* F. Morley, Lectures 1915-16, Johns Hopkins University.

PART III.

CONCOMITANTS IN METRICAL GEOMETRY.

Coming now to the realm of metrical geometry, we will derive a complete system of curves symmetrically related to the members of a point-set. We assume barycentric coordinates, the line at infinity, and the two imaginary circular points (the absolute) on this line.

Section (a): The Three-Point.

Take the three points as vertices of the reference triangle, and denote by λ_i the *square* of the length of side lying on $x_i=0$. Then, any curve connected metrically *and symmetrically* to the points is expressed as a function of two rows,

$$\begin{matrix} x_0, & x_1, & x_2, \\ \lambda_0, & \lambda_1, & \lambda_2, \end{matrix}$$

which remains unchanged, to within the algebraic sign, when λ_i and x_i are interchanged with λ_j and x_j, respectively.

Thus, our problem reduces to that of deriving a complete system of such functions.

Section (b): The Four-Point.

If the four points are referred to their diagonal triangle as reference triangle, their coordinates are of the form

$$b_0, \ \pm b_1, \ \pm b_2,$$

and the entire set is defined by three quantities a_0, a_1, a_2, such that $a_i=b_i^2$. Then a curve of the kind desired is expressed as a function of three rows,

$$\begin{matrix} x_0, & x_1, & x_2, \\ \lambda_0, & \lambda_1, & \lambda_2, \\ a_0, & a_1, & a_2, \end{matrix}$$

which remains unchanged, to within the algebraic sign, when x_i, λ_i, a_i, are interchanged with x_j, λ_j, a_j, respectively.

We must derive, then, a complete system of these functions.

Since the three-rowed functions include, as a special case, those of two rows, one algebraic investigation will suffice to determine the complete systems of both classes of functions.

The functions are classified as (1) symmetric, (2) alternating, according as the algebraic sign (1) is not, (2) is, changed when a permutation of elements is made.

Section (c): Symmetric Functions.

The theory of symmetric functions of one row has been thoroughly developed.* The theory for several rows has been investigated by Junker and MacMahon. To some of the articles of the former † reference has been made in the preparation of the following paragraphs:

Symmetric functions of the kind under consideration will be of the form

$$J=\sum^{6}\lambda_0^{\alpha}\lambda_1^{\alpha'}\lambda_2^{\alpha''}a_0^{\beta}a_1^{\beta'}a_2^{\beta''}x_0^{\gamma}x_1^{\gamma'}x_2^{\gamma''},$$

the exponents being positive integers or zero. In certain cases (for instance, when $\alpha=\alpha'$, $\beta=\beta'$, $\gamma=\gamma'$) the function has only three distinct terms. It will then be indicated by the symbol $\overset{3}{\Sigma}$, or by placing its characteristic term in round brackets. Attention is called also to the products $\lambda_0\lambda_1\lambda_2$, $a_0a_1a_2$, $x_0x_1x_2$, which are symmetric functions of *one term only.*

We will prove that J can be expressed integrally in terms of three-term functions.

Let $\quad p_i=\lambda_i^{\alpha}a_i^{\beta}x_i^{\gamma},\quad q_i=\lambda_i^{\alpha'}a_i^{\beta'}x_i^{\gamma'},\quad r_i=\lambda_i^{\alpha''}a_i^{\beta''}x_i^{\gamma''}.$

Then
$$\begin{aligned}J&=\sum^{3}p_0(q_1r_2+q_2r_1)\\&=\sum^{3}p_0\{\sum^{3}q_0(r_1+r_2)-[q_0(r_1+r_2)+r_0(q_1+q_2)]\}\\&=\sum^{3}p_0\{\sum^{3}q_0((r)-r_0)-[q_0((r)-r_0)+r_0((q)-q_0)]\}\\&=\sum^{3}p_0[(q)(r)-(qr)-(r)q_0+2q_0r_0-(q)r_0]\\&=(p)(q)(r)-(p)(qr)-(pq)(r)+2(pqr)-(q)(pr).\end{aligned}$$

Therefore

(A) *The complete system will contain no six-term symmetric functions.*

Since J has as a factor the symmetric function $\lambda_0^m\lambda_1^m\lambda_2^m$, where m is the smallest of the α's (similarly for a and x):

(B) *The complete system will contain no symmetric functions with more than two λ's, two a's and two x's in a term.*

Consider
$$\begin{aligned}(Ax^n)\ &=A_0x_0^n\ +A_1x_1^n\ +A_2x_2^n,\\(Ax^{n-1})&=A_0x_0^{n-1}+A_1x_1^{n-1}+\ldots,\\(Ax^{n-2})&=A_0x_0^{n-2}+\ldots,\\(Ax^{n-3})&=A_0x_0^{n-3}+\ldots,\end{aligned}$$

* Salmon, "Higher Algebra."

† *Math. Ann.*, Vols. XLIII, XLV.

where the A's are expressions in λ, a, and x, consistent with the three-term form in which used.

Eliminating the A's, we obtain

$$\begin{vmatrix} (Ax^n) & x_0^n & x_1^n & x_2^n \\ (Ax^{n-1}) & x_0^{n-1} & x_1^{n-1} & x_2^{n-1} \\ (Ax^{n-2}) & x_0^{n-2} & x_1^{n-2} & x_2^{n-2} \\ (Ax^{n-3}) & x_0^{n-3} & x_1^{n-3} & x_2^{n-3} \end{vmatrix} = 0,$$

which, after the removal of certain factors, gives

$$(Ax^n) = (x)(Ax^{n-1}) - (x_1x_2)(Ax^{n-2}) + x_0x_1x_2(Ax^{n-3}).$$

Repeating the process on (Ax^{n-1}), etc., we obtain finally (Ax^n) in terms of (Ax^2), (Ax), (A), (x), (x_1x_2) and $x_0x_1x_2$. Since the process applies equally well to the λ's and the a's.

(C) *The complete system will contain no symmetric functions in which an element occurs to a higher degree than the second.*

We state for reference the following special case of the above formula:

$$(Ax^3) = (x)(Ax^2) - (x_1x_2)(Ax) + x_0x_1x_2(A). \tag{1}$$

Since by (B) no term is to contain more than two elements of a kind, the total degree in x (λ or a) must not exceed four. But, if the function be $(A_0x_1^2x_2^2)$, of total degree 4 in x, we can use the important identities

$$x_ix_j = (x_1x_2) - (x)x_k + x_k^2 \qquad (i, j, k = 0, 1, 2;\ i \neq j \neq k), \tag{2}$$

and obtain

$$(A_0x_1^2x_2^2) = (A)(x_1x_2)^2 + (x)^2(Ax^2) + (Ax^4) - 2(x)(x_1x_2)(Ax) \\ - 2(x_1x_2)(Ax^2) - 2(x)(Ax^3),$$

in which (Ax^4) and (Ax^3) are reducible by (C).

A function of total degree 3 in x is of the type $\Sigma A_0x_1^2x_2$. This necessarily contains six terms and is therefore excluded by (A). Hence:

(D) *The complete system will contain no symmetric functions in which the total degree in λ, a, or x is greater than* 2, *except the special functions* $\lambda_0\lambda_1\lambda_2$, $a_0a_1a_2$, $x_0x_1x_2$.

For reference, we restate (2) in different form:

$$x_i^2 = x_jx_k + (x)x_i - (x_1x_2) \qquad (i, j, k = 0, 1, 2;\ i \neq j \neq k). \tag{3}$$

A function can contain x to total degree 2 in two ways:

$$(Ax^2) \quad \text{or} \quad (A_0x_1x_2).$$

But, by (2),

$$(A_0x_1x_2) = (A)(x_1x_2) - (x)(Ax) + (Ax^2).$$

Hence:

(E) *As members of the complete system of symmetric functions, the forms* $(A_0x_1x_2)$ *and* (Ax^2) *(similarly for the* λ*'s and* a*'s) are mutually exclusive.*

We will adopt the form $(A_0x_1x_2)$.

Denoting by s_{pqr} a three-term symmetric function of total degree p in λ, q in a, and r in x; principles (A) to (E), inclusive, leave for individual consideration

s_{001}	s_{010}	s_{100}
s_{002}	s_{020}	s_{200}
s_{003}	s_{030}	s_{300}
s_{011}	s_{101}	s_{110}
s_{012}	s_{102}	s_{120}
s_{021}	s_{201}	s_{210}
s_{022}	s_{202}	s_{220}
s_{111}		
s_{112}	s_{121}	s_{211}
s_{122}	s_{212}	s_{221}
s_{222}		

We need consider only the first column in detail, because the others are obtained from it by interchange of letters.

$s_{001}=(x)$ and $s_{002}=(x_1x_2)$ are obviously irreducible.

$s_{003}=x_0x_1x_2$, $s_{011}=(ax)$, $s_{012}=(a_0x_1x_2)$, and $s_{021}=(a_1a_2x_0)$ are found to be irreducible by a test which will be illustrated by use on $s_{022}=(a_1a_2x_1x_2)$.

This, if reducible, will be a sum of products of irreducible functions, each product of degree 022. Assume then

$$(a_1a_2x_1x_2)=k(a)(a_0x_1x_2)+l(x)(a_1a_2x_0)+m(a_1a_2)(x_1x_2)+n(ax)^2$$
$$+p(a)^2(x_1x_2)+q(x)^2(a_1a_2)+r(a)(x)(ax)+s(a)^2(x)^2.$$

Then, by substituting sets of numerical values for the a's, and x's we obtain a sufficient number of equations, simultaneous in k, l, m, etc., to solve for their values. The operation is greatly simplified by a selection of numbers which causes several terms of the assumed identity to vanish.

For instance, substituting

$$1,\ -1,\ 0;\ \ 1,\ 1,\ -2 \quad \text{for} \quad a_0,\ a_1,\ a_2;\ \ x_0,\ x_1,\ x_2,$$

respectively; (a), (ax), and (x) vanish, and we obtain

$$-1=3m, \quad \text{whence} \quad m=-1/3.$$

Similarly, from

$$1, -1, 0;\ 1, -1, 0;\quad n=1/3,$$
$$1, \ \ 0, 0;\ 1, -1, 0;\quad p=1/3, \text{ etc.}$$

As a result

$$\text{(F)}\quad s_{022}=(a_1a_2x_1x_2)=-1/3\,(a)\,(a_0x_1x_2)-1/3\,(x)\,(a_1a_2x_0) -1/3\,(a_1a_2)(x_1x_2)+1/3\,(ax)^2 +1/3\,(a)^2(x_1x_2)+1/3\,(x)^2(a_1a_2)-1/3\,(a)(x)(ax),$$

the correctness of which has been tested by various numerical substitutions.

If such substitutions fail to satisfy the expression, or if contradictions arise in solving for the coefficients, the assumption of an identity is false and the function is irreducible. As an illustration, assume

$$s_{012}=(a_0x_1x_2)=k\,(x)\,(ax)+l\,(a)\,(x_1x_2)+m\,(a)\,(x)^2.$$

From $1, -1, 0;\ 1,\ \ 0, 0;\quad k=0,$

and from $1,\ \ 1, 0;\ 1, -1, 0;\quad l=0,$

whence $(a_0x_1x_2)=m\,(a)\,(x)^2,$

which is obviously untrue, whatever the value of m.

In this way s_{003}, s_{011}, s_{021} and $s_{111}=(\lambda ax)$ are proved irreducible. Similar methods show that

$$\text{(G)}\quad s_{112}=(\lambda_0a_0x_1x_2)=1/3\,(\lambda)\,(a_0x_1x_2)+1/3\,(a)\,(\lambda_0x_1x_2) -1/3\,(x)\,(\lambda ax)+2/3\,(\lambda a)\,(x_1x_2) +1/3\,(\lambda x)(ax)-1/3\,(\lambda)\,(a)\,(x_1x_2),$$

which stands the test for an identity.

(H) $s_{122}=(\lambda_0a_1a_2x_1x_2)$ reduces by substituting a_1a_2 for a_0, a_2a_0 for a_1, a_0a_1 for a_2 in (G). Similarly,

(K) $s_{222}=(\lambda_1\lambda_2a_1a_2x_1x_2)$ reduces by substituting $\lambda_1\lambda_2$ for λ_0, etc., in (H).

Complete System.—We have, therefore, the following complete system of symmetric functions:

Of λ alone	s_{100}	s_{200}	s_{300}
Of a alone	s_{010}	s_{020}	s_{030}
Of x alone	s_{001}	s_{002}	s_{003}
Of a and x	s_{011}	s_{012}	s_{021}
Of λ and x	s_{101}	s_{102}	s_{201}
Of λ and a	s_{110}	s_{120}	s_{210}
Of λ, a and x	s_{111}		

Furthermore, the foregoing processes, (A) to (K) inclusive, enable us to express any given symmetric function in terms of the members of the complete system.

Section (d): Alternating Functions.

Using the same notation as in Section (c), the most general alternating function of three rows is

$$K = p_0q_1r_2 + p_1q_2r_0 + p_2q_0r_1 - p_0q_2r_1 - p_1q_0r_2 - p_2q_1r_0 .$$

This is the same as

$$|pqr| \equiv \begin{vmatrix} p_0 & q_0 & r_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} .$$

Taking K however in the form $\overset{3}{\Sigma} Ap$, where $A_0 = q_1r_2 - q_2r_1$, etc., it is easily proved that

(B′) *The complete system will contain no alternating functions with more than two λ's, two a's and two x's in a term.*

(C′) *The complete system will contain no alternating functions in which an element occurs to a degree higher than the second.*

Functions containing λ, a or x to total degree 4 can be reduced by (2), Section (c), to functions of total degree 2 in that quantity. Functions of total degree 3 in λ, a or x are not reducible by this means. Hence

(D″) *The complete system will contain no alternating functions in which the total degree in λ, a or x is greater than* 3.

(E) can be restated:

(E′) *As members of the complete system,* $\overset{3}{\Sigma} A_i x_1 x_2$ *and* $\overset{3}{\Sigma} A_i x_0^2$ *(similarly for the λ's and a's) are mutually exclusive.*

A further important principle is obtained by considering the determinant

$$\begin{vmatrix} p_0 & q_0 & r_0 & 1/3 \\ p_1 & q_1 & r_1 & 1/3 \\ p_2 & q_2 & r_2 & 1/3 \\ (p) & (q) & (r) & 1 \end{vmatrix} .$$

This vanishes identically because the last row is the sum of the others. Hence, expanding:

$$|p\ q\ r| = 1/3(p)\,|q\ r\ 1| - 1/3(q)\,|p\ r\ 1| + 1/3(r)\,|p\ q\ 1| .$$

This reduces any alternating function except those in which p, q or r equals unity. Therefore:

(L) *The complete system will contain no alternating functions except those whose determinants have a column (or row) of 1's.*

Such functions, of total degree 3 in x, are of the type

$$|ux^2\ vx\ 1|$$

where u and v are products of powers of λ and a. By (2), Section (c),

$$|ux^2\ vx\ 1| = (x_1x_2)\,|ux\ v\ 1| - (x)\,|ux\ v\ x| + |ux\ v\ x^2|.$$

Only the last function in this expansion contains x to total degree as high as 3. Applying (L) to this function,

$$|ux\ v\ x^2| = 1/3\,(ux)\,|v\ x^2\ 1| - 1/3\,(v)\,|ux\ x^2\ 1| + 1/3\,(x^2)\,|ux\ v\ 1|.$$

Likewise, in this expansion, only $|ux\ x^2\ 1|$ is of total degree 3 in x. Applying (2) to this,

$$|ux\ x^2\ 1| = (x_1x_2)\,|u\ x\ 1| + |u\ x\ x^2|.$$

Again, applying (L) to $|u\ x\ x^2|$,

$$|u\ x\ x^2| = (u)\,|1\ x\ x^2| - (x)\,|u\ x^2\ 1| + (x^2)\,|u\ x\ 1|.$$

Finally, $|1\ x\ x^2|$ is unaltered by (2), (3) or (L).

If u, v, or both equal unity, the process is merely shortened; the conclusion is the same, namely:

(D') *The complete system will contain no alternating functions in which the total degree in λ, a or x is greater than 2, except possibly $|x^2\ x\ 1|$, $|\lambda^2\ \lambda\ 1|$ and $|a^2\ a\ 1|$.*

Consider the cases where at least two elements occur, each to total degree 2. If w and w' are powers of λ, we have the types

(a) $|wax\ \ w'ax\ \ 1|$, (b) $|wax^2\ \ w'a\ \ 1|$, (c) $|wa^2x^2\ \ w'\ \ 1|$,
(d) $|wa^2x\ \ w'x\ \ 1|$, (e) $|wa^2\ \ w'x^2\ \ 1|$.

By (2), (a) reduces to $|wa\ \ w'a\ \ x^2|$ and functions of lower degree.
By (3), (b) " " $|wa\ \ w'ax\ \ x|$ " " " " "
By (3), (c) " " $|wa^2\ \ w'x\ \ x|$ " " " " "
By (2), in terms of the element a,
(d) reduces to $|wx\ \ w'ax\ \ a|$ " " " " "
By (3), (e) " " $|wa^2x\ \ w'\ \ x|$ " " " " "

All of these results can be reduced by (L), except that of (e) when $w'=1$, thus giving $|wa^2x\ x\ 1|$. But, by (3) in terms of the element a, this reduces to $|wx\ ax\ a|$ (and functions of lower degree) and (L) reduces this form. Hence:

(M) *The complete system will contain no alternating functions in which more than one element occurs to total degree* 2.

We have remaining, for individual consideration, the types:

a_{001}	a_{010}	a_{100}
a_{002}	a_{020}	a_{200}
a_{003}	a_{030}	a_{300}
a_{011}	a_{101}	a_{110}
a_{012}	a_{102}	a_{120}
a_{021}	a_{201}	a_{210}
a_{111}		
a_{112}	a_{121}	a_{211}

a_{001} and a_{002} can be constructed only in forms which vanish identically. a_{003} has, by (D′), the one type, $|x^2\ x\ 1|$, which is obviously irreducible. a_{011} can appear only as $|x\ a\ 1|$ and is obviously irreducible. a_{012} appears either as $|x^2\ a\ 1|$ or $|x\ ax\ 1|$. The only possible assumption is

$$|x^2\ a\ 1| \text{ or } |x\ ax\ 1| = k(x)\,|x\ a\ 1|.$$

If $a_i = x_i$, we have

$$|x^2\ x\ 1| \text{ or } |x\ x^2\ 1| = k(x)\,|x\ x\ 1| = 0,$$

which is untrue. By (2),

$$|x\ ax\ 1| = |1\ a\ x^2| - (x)\,|1\ a\ x| - |x^2\ a\ 1| + (x)\,|x\ a\ 1|.$$

Hence, $|x^2\ a\ 1|$ can be taken as the irreducible type.

Interchanging a and x in this form, $a_{021} = |x\ a^2\ 1|$ is seen to be irreducible. a_{111} can appear as:

$$\text{(a)}\quad |x\ \ \lambda a\ \ 1| = \sum^{3} x_0(\lambda_1 a_1 - \lambda_2 a_2),$$

$$\text{(b)}\quad |\lambda x\ \ a\ \ 1| = \sum^{3} \lambda_0 x_0 (a_1 - a_2),$$

$$\text{(c)}\quad |ax\ \ \lambda\ \ 1| = \sum^{3} a_0 x_0 (\lambda_1 - \lambda_2).$$

The sum of (a) and (b) is

$$\begin{aligned}\sum^{3} x_0[a_1(\lambda_1+\lambda_0) - a_2(\lambda_2+\lambda_0)] &= \sum^{3} x_0[a_1((\lambda)-\lambda_2) - a_2((\lambda)-\lambda_1)]\\ &= (\lambda)\sum^{3} x_0(a_1-a_2) - \sum^{3} x_0(\lambda_2 a_1 - \lambda_1 a_2)\\ &= (\lambda)\,|x\ a\ 1| - |x\ a\ \lambda|.\end{aligned}$$

Hence

$$|\lambda x\ a\ 1| = -|x\ \lambda a\ 1| + (\lambda)\,|x\ a\ 1| - |x\ a\ \lambda|,$$

the last term of which is reducible by (L).

Interchanging λ and a,

$$|ax\,\lambda\,1| = -|x\,\lambda a\,1| + (a)\,|x\,\lambda\,1| - |x\,\lambda\,a|.$$

Thus, (b) and (c) are expressible in terms of (a) and functions of lower degree.

Assume (a):

$$|x\,\lambda a\,1| = k(x)\,|\lambda\,a\,1| + l(a)\,|x\,\lambda\,1| + m(\lambda)\,|x\,a\,1|.$$

Let $\lambda_i = x_i$, $a_i = x_i$, then

$$|x\,x^2\,1| = k(x)\,|x\,x\,1| + l(x)\,|x\,x\,1| + m(x)\,|x\,x\,1| = 0,$$

which is untrue. Hence

$a_{111} = |x\,\lambda a\,1|$ is an irreducible function.

a_{112} can appear as

(a) $|x^2\;\lambda a\;1|$, (b) $|\lambda ax\;x\;1|$, (c) $|\lambda x^2\;a\;1|$
(d) $|ax^2\;\lambda\;1|$, (e) $|\lambda x\;ax\;1|$.

By (2), (b) reduces to (a) and functions of lower degree.
By (3), (c) " " $|\lambda\;ax\;x\,|$ and functions of lower degree.
By (3), (d) " " $|a\;\lambda x\;x\,|$ " " " " "
By (2), (e) " " $|\lambda\;a\;x^2|$ " " " " "

Hence (a) is the one form requiring further examination. Adding this to (c), we obtain finally

(N) $$|x^2\,\lambda a\,1| = -|\lambda x^2\,a\,1| + (\lambda)\,|x^2\,a\,1| - |x^2\,a\,\lambda|,$$

and the right-hand members are reducible.

Complete System.—We have, therefore, the following complete system of alternating functions:

Of λ alone	$a_{300} = \lvert\lambda^2\;\lambda\;1\rvert$.		
Of a alone	$a_{030} = \lvert a^2\;a\;1\rvert$.		
Of x alone	$a_{003} = \lvert x^2\;x\;1\rvert$.		
Of a and x	$a_{011} = \lvert x\;a\;1\rvert$,	$a_{012} = \lvert x^2\;a\;1\rvert$,	$a_{021} = \lvert x\;a^2\;1\rvert$.
Of λ and x	$a_{101} = \lvert x\;\lambda\;1\rvert$,	$a_{102} = \lvert x^2\;\lambda\;1\rvert$,	$a_{201} = \lvert x\;\lambda^2\;1\rvert$.
Of λ and a	$a_{110} = \lvert\lambda\;a\;1\rvert$,	$a_{120} = \lvert\lambda\;a^2\;1\rvert$,	$a_{210} = \lvert\lambda^2\;a\;1\rvert$.
Of λ, a and x	$a_{111} = \lvert x\;\lambda a\;1\rvert$.		

Furthermore, the foregoing processes enable us to express any given alternating function in terms of the members of the complete systems of symmetric and of alternating functions.

Section (e): Functions of n Rows of Elements.

The operations explained in Sections (c) and (d) are not confined to functions of three rows only. It is easily seen that theorems (A) to (E) inclusive and (A′) to (M) inclusive can be re-stated, without material alteration, for a function of n rows of three quantities each.

Consider a symmetric function of four rows, β, λ, a, x. The complete system will contain the irreducible functions of λ, a, x, and similar functions of β, a, x, etc. In addition, we must examine

$$\begin{array}{ll} s_{1111} & \\ s_{1112} & s_{1121}, \text{ etc.,} \\ s_{1122} & s_{1212}, \text{ etc.,} \\ s_{1222}, \text{ etc.,} & \\ s_{2222} & \end{array}$$

$s_{1112}=(\beta_0\lambda_0 a_0 x_1 x_2)$ reduces by substituting $\beta_i\lambda_i$ for λ_i in (G). Similar methods apply to s_{1122}, s_{1222}, s_{2222}. Assume

$$s_{1111}=(\beta\lambda ax)=k(\beta)(\lambda ax)+m(\lambda)(\beta ax)+\ldots\ldots$$

Setting $\beta_0=\beta_1=\beta_2=1$, and collecting terms, we have

$$(\lambda ax)=k'(\lambda ax)+m'(\lambda)(ax)+n'(a)(\lambda x)+p'(x)(\lambda a)+q'(\lambda)(a)(x).$$

If $k'\neq 1$, this gives an expansion for (λax), which is impossible. If $k'=1$, we have

$$m'(\lambda)(ax)+n'(a)(\lambda x)+p'(x)(\lambda a)+q'(\lambda)(a)(x)=0.$$

Let $\lambda_0=a_0=1$, $\lambda_1=a_1=-1$, $\lambda_2=a_2=0$, so that $(\lambda)=(a)=0$. Then

$$2p'(x)=0, \text{ whence } p'=0.$$

Similarly, $m'=n'=q'=0$, so the identity is not true for finite coefficients. Therefore:

Of symmetric functions of finite degree in all four rows, s_{1111} alone is irreducible.

Likewise

$a_{1111}=|\beta\lambda a\ x\ 1|$ *is the only irreducible alternating function of finite degree in each of four rows.*

Similar statements are readily seen to hold true for 5, 6, ..., n rows.

Section (f): Geometric Classification.

The Three-Point.—The complete system for the three-point consists of those forms, deduced in Sections (*c*) and (*d*), which do not contain a. The subscripts denoting degrees in λ and x, respectively, we have

	Symmetric	Alternating	Total
Cubics	s_{03}	a_{03}	2
Conics	$s_{02}\ s_{12}$	a_{12}	3
Lines	$s_{01}\ s_{11}\ s_{21}$	$a_{11}\ a_{21}$	5
Invariants	$s_{10}\ s_{20}\ s_{30}$	a_{30}	4
Total	9	5	14

The Four-Point.—The complete system for the four-point consists of the entire list obtained in Sections (*c*) and (*d*). Rearranged according to degree in x, they are:

	Symmetric	Alternating	Total
Cubics	s_{003}	a_{003}	2
Conics	$s_{002}\ s_{012}\ s_{102}$	$a_{012}\ a_{102}$	5
Lines	$s_{001}\ s_{011}\ s_{021}\ s_{201}\ s_{101}\ s_{111}$	$a_{011}\ a_{101}\ a_{111}\ a_{021}\ a_{201}$	11
Invariants	$s_{100}\ s_{010}\ s_{110}\ s_{210}\ s_{120}\ s_{020}\ s_{200}$ $s_{030}\ s_{300}$	$a_{300}\ a_{030}\ a_{120}\ a_{210}\ a_{110}$	14
Total	19	13	32

The members of the two systems are the concomitants of the three-point and the four-point under linear transformations sending the line at infinity into itself, either leaving the two points of the absolute fixed, or interchanging them.

Section (g): Syzygies.

It is not our purpose to derive a system of syzygies. However, it appears that the following method is useful in doing so. Take the determinant

$$\begin{vmatrix} (x^2) & x_0^2 & x_1^2 & x_2^2 \\ (\lambda x) & \lambda_0 x_0 & \lambda_1 x_1 & \lambda_2 x_2 \\ (ax) & a_0 x_0 & a_1 x_1 & a_2 x_2 \\ (x) & x_0 & x_1 & x_2 \end{vmatrix} = 0,$$

which vanishes identically because the first column is the sum of the others. Removing the factor $x_0 x_1 x_2$ and expanding:

$$(x^2)\,|\lambda\ a\ 1| - (\lambda x)\,|x\ a\ 1| + (ax)\,|x\ \lambda\ 1| - (x)\,|x\ \lambda\ a| = 0.$$

Substituting $(x)^2-2(x_1x_2)$ for (x^2), applying (L) to $|x\,\lambda\,a|$ and clearing of fractions, we obtain, in the abridged notation,

$$(2s_{001}^2-6s_{002})a_{110}+(s_{100}\,s_{001}-3s_{101})a_{011}+(3s_{011}-s_{010}\,s_{001})a_{101}=0.$$

Section (*h*): *Geometric Interpretation.*

In the following we shall denote $\sqrt{\lambda_i}$ by l_i, and $\sqrt{a_i}$ by b_i. Then, it will be remembered, l is the length of a side of the reference triangle, and b_0, $\pm b_1$, $\pm b_2$, are the coordinates of four points having the reference triangle as their diagonal triangle. So far, we have considered the a's, λ's and l's merely as magnitudes, but, of course, they are also the coordinates of certain points, the positions of which it is well to establish. A point h_0, h_1, h_2, will be denoted by the symbol h.

It is known that

1 is the centroid of the triangle,

l is the incenter,

λ is the symmedian point—the center of perspective of the vertices of the triangle with the intersections of tangents to the circumcircle at the vertices.

$1/\lambda$: Rays from a vertex through λ and $1/\lambda$ meet the opposite side in points equidistant from its mid-point.

To locate a:

Given points b_0, $\pm b_1$, $\pm b_2$, the lines joining them, two at a time, are on the vertices of the triangle—two lines on each vertex—and form a harmonic pencil with the two sides of the triangle on the same vertex. Two such lines, on 1, 0, 0, are

$$b_2x_1-b_1x_2=0, \text{ and } b_2x_1+b_1x_2=0,$$

which, taken together, form a degenerate conic

$$b_2^2x_1^2-b_1^2x_2^2=a_2x_1^2-a_1x_2^2=0.$$

The polar line of this with respect to 1 is

$$a_2x_1-a_1x_2=0,$$

on which lies a.

From the nature of this construction, the line on 1, 0, 0 and 1, (the median) is the harmonic conjugate of $a_2x_1-a_1x_2=0$ with respect to the two lines of the degenerate conic.

Hence, given a point b, to construct point a:

Join b to a vertex of the reference triangle by a line m and take n, the fourth harmonic of m with respect to the sides of the triangle on that vertex.

Take the fourth harmonic p of the median with respect to m and n. Performing this construction for each vertex, the three lines p will meet in a.

λ^2: By the above rule λ^2 is obtainable from λ.

It is interesting to note that, if b_0, $\pm b_1$, $\pm b_2$ are the coordinates of four *lines*, the *line* a is on the mid-points of the diagonals of the four-line configuration.

λa: Taking the polar of the degenerate conic $a_2x_1^2-a_1x_2^2=0$, with respect to $1/\lambda$, we have

$$\lambda_2a_2x_1-\lambda_1a_1x_2=0,$$

on which lies λa. The details of the construction readily follow.

If we calculate the polar systems of $x_0x_1x_2=0$, (the triangle itself) and $|x^2\ x\ 1|=(x_1-x_0)(x_1-x_2)(x_2-x_0)=0$ (the three medians) with respect to the various points defined above, it is seen that they are closely identified with the complete system of Section (f). In several cases, the forms of the complete system and of the polar system are identical, and, most generally, they are members of the same pencil. Hence, the geometrical construction of the complete system can be based upon the construction of the polar system.

The expanded form of $|x^2\ x\ 1|$ in the preceding paragraph shows it to be the discriminant of $x_0x_1x_2=0$. Similarly for $|\lambda^2\ \lambda\ 1|$ and $|a^2\ a\ 1|$.

Systems of Pencils of Lines in Ordinary Space.

By Alton L. Miller.

In his classic paper entitled "Preliminari di una Teoria delle Varietá Luoghi di Spazi," Segre* laid the foundation of investigations of the projective differential properties of geometric configurations in n-dimensions by synthetic methods. It is the purpose of this paper to apply some of the results of these investigations to the study of families of pencils of lines in ordinary space. By means of the Klein coordinates of a line, there is set up a one to one correspondence between lines of space and points of a hyperquadric, Q, in five dimensions. To a line on this hyperquadric there corresponds in S_3 a pencil of lines. Thus, to ruled varieties on Q there correspond families of pencils in S_3. This correspondence is discussed completely in the following paper.

Let $(x)=(x_0, x_1, x_2, x_3, x_4, x_5)$ be the Klein coordinates of a line in ordinary space, then

$$(xx)=\sum_0^5 x_i^2=0. \tag{1}$$

If at the same time we think of (x) as a point in five dimensions, equation (1) represents a hyperquadric, Q, in that space. Thus, to every line in ordinary space there corresponds in five dimensions a point on the hyperquadric, Q.

If two lines intersect, their coordinates satisfy the relation $(yz)=\sum_0^5 z_i y_i=0$, and the coordinates of any line in their pencil are given by $(x)=\lambda(y)+\mu(z)$ for some value of $\lambda:\mu$. The points of Q which correspond to two intersecting lines are therefore conjugate with respect to Q and every point $(x)=\lambda(y)+\mu(z)$ lies on Q, on the line from (y) to (z).

Part I.

1. A ruled surface, R, on Q corresponds in S_3 to a one-parameter family of pencils of lines, **R**. The centers of these pencils, in general, trace a curve C, while their planes, in general, envelop a developable, c. If R is developable,

* *Rendiconti del Circolo Matematico di Palermo*, Vol. XXX (1910).

C is the edge of regression of c. Points on the edge of regression of R correspond to the characteristics of c. If R lies in an S_4 the corresponding **R** lies in a linear complex, and if R lies in an S_3 **R** belongs to a linear congruence. This congruence will have distinct, coplanar, or coincident directrices according as R lies in an S_3 which cuts Q in a non-degenerate quadric, a pair of planes, or a cone.

PART II.

2. A two-parameter family of lines on Q generates a ruled V_3 which we will call M. To M there corresponds in S_3 a two-parameter family of pencils generating a line complex which we will call an m-complex. Every linear complex is an m-complex. In general the centers of the ∞^2 pencils trace a surface, S, and their planes envelop a non-developable surface, s. S may reduce to a curve or even a point, and, in the dual case, s may reduce to a developable or even a plane. In general S and s do not coincide.

3. We can represent M analytically as follows: Let $x_i = \alpha_i(u, v)$ and $x_i = \beta_i(u, v)$ $i = 0, 1, \ldots, 5$ be any two surfaces on Q. That is $(\alpha\alpha) = (\beta\beta) = 0$ for all values of u, v. Furthermore, let us assume that if A and B are two corresponding points of α and β, that is two points obtained by giving u and v the same values in α and β, then the line from A to B lies entirely on Q. The necessary and sufficient condition for this is

$$(\alpha\beta) = 0 \text{ for all values of } u \text{ and } v. \tag{2}$$

Then lines joining corresponding points of α and β generate a ruled V_3 of Q, M. Hence M is given by

$$x_i = \alpha_i(u, v) + t\beta_i(u, v) \qquad i = 0, 1, \ldots, 5. \tag{3}$$

If we fix u and v in (3) but cause t to vary (x) traces a line generator of M.

In S_3 the α and β surfaces represent line congruences of such a nature that corresponding lines of each intersect. If we fix u and v in (3), but cause t to vary, (x) traces a pencil of lines, one of the generating pencils of the m-complex represented by (3).

The tangent S_3 to M at (x_0), *i. e.*, $u = u_0$, $v = v_0$, $t = t_0$, is determined by (x), (x_u), (x_v), (x_t), that is by the four points

$$(\alpha), \quad (\beta), \quad (\alpha_u + t\beta_u), \quad (\alpha_v + t\beta_v). \tag{4}$$

Let G represent the generator from α to β. If we allow t to vary in (4), but hold u and v fast, (4) determines the various tangent S_3's to M at points of G.

We will call the linear congruence corresponding to the tangent S_3 to M, *tangent* to the m-complex at the line corresponding to the point of contact of

the S_3. Such a tangent linear congruence will always be degenerate in that its directrices will be coincident. It will be determined by a line of m and three lines of m infinitely near the first, but not lying in the same regulus with the first. For example by (x), $(x)+(x_u)\Delta u$, $(x)+(x_v)\Delta v$, $(x)+(x_t)\Delta t$, where Δu, Δv, Δt are all infinitesimal. Evidently the S_3 in S_5 determined by these four points is precisely the same as the S_3 determined by the points of (4), although those points do not lie on Q and therefore do not correspond to lines in ordinary space. All of the linear congruences tangent to m at lines of the pencil g have g in common, but in general, nothing else.

4. A line of M infinitely near G will join a point of the α-surface infinitely near (α) to a point of the β-surface infinitely near (β). That is a line of M infinitely near G will be determined by $(\alpha+\alpha_u\Delta u+\alpha_v\Delta v)$, $(\beta+\beta_u\Delta u+\beta_v\Delta v)$, where Δu and Δv are infinitesimal. As we vary the ratio of $\Delta u:\Delta v$ we get ∞^1 lines of M infinitely near G. G and a line of M infinitely near G determine the S_3 of the points

$$(\alpha),\quad (\beta),\quad (\alpha_u\Delta u+\alpha_v\Delta v),\quad (\beta_u\Delta u+\beta_v\Delta v). \tag{5}$$

Segre* has shown that the locus of the ∞^1 S_3's obtained by varying the ratio $\Delta u:\Delta v$ in (5) is a quadratic cone in S_5. This quadratic cone has G for a line of double points, and one other double point in each plane of Q that contains G exists for the quartic of intersection of Q with this cone. Hence the intersection of Q with the locus of all S_3's determined by G and the lines of M infinitely near G, is a quartic with double points along G and one in each of the planes of Q that contain G.

The above theorem stated in terms of line geometry becomes:

The locus of the ∞^1 linear congruences determined by a pencil g of an m-complex, and the pencils of the m-complex infinitely near g is a quadratic complex with double lines, the lines of g and one other in the plane that contains g and one other through the center of g. This is a tetrahedral complex of the type [(22)(11)].†

5. If two of the tangent S_3's at points of a generator G have a plane in common, all the S_3's tangent to M at points of G lie in an S_4. For the S_3 tangent to M at P_1 of G is the S_3 of (α), (β), $(\alpha_u+t_1\beta_u)$, $(\alpha_v+t_1\beta_v)$, and that at P_2 of (α), (β), $(\alpha_u+t_2\beta_u)$, $(\alpha_v+t_2\beta_v)$. Since these two have a plane in common, by Grassman's theorem, they lie in an S_4, and any linear combina-

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* Segre, *loc. cit.*, No. 12.

† A. Weiler, *Math. Annalen*, Vol. VII (1874). Sturm, "Liniengeometrie." III. p. 436.

tion of the above eight points lies in the same S_4. In particular the following points lie in an S_4:

$$(\alpha), \quad (\beta), \quad (\alpha_u), \quad (\beta_u), \quad (\alpha_v), \quad (\beta_v). \tag{6}$$

Hence every tangent S_3 to M at a point of G lies in this S_4. Note that it does not follow that all tangent S_3's to M at points of G have a plane in common. This is not in general true.

In line space the above theorem states: *If two of the tangent linear congruences to m at lines of a pencil g have in common besides g a second pencil h having a line in common with g, all the tangent linear congruences to m at lines of g lie in a linear complex.*

The S_3's tangent to M along a generator G all have a plane in common if, and only if, the S_3's determined by G and the lines of M infinitely near G have a plane in common. In fact the tangent S_3's of (4) are S_3's determined by G and lines of the regulus obtained when we vary t in the line from $(\alpha_u+t\beta_u)$ to $(\alpha_v+t\beta_v)$. If a plane is to be common to all these S_3's it must cut every line of this regulus. Hence the regulus cuts that plane either in a conic, which is impossible, for then G would cut the regulus twice, and all the tangents S_3's would coincide, or else in a line. Then G cuts the regulus in one point. Hence a necessary and sufficient condition that all the tangent S_3's to M along G have a plane in common is that G cut the regulus mentioned above.

But the S_3's determined by G and the lines infinitely near G are the same S_3's as those determined by G, and the lines of the regulus obtained by varying the ratio $\Delta u:\Delta v$ in the line from $(\alpha_u\Delta u+\alpha_v\Delta v)$ to $(\beta_u\Delta u+\beta_v\Delta v)$. And a necessary and sufficient condition that all these S_3's have a plane in common is that G cut this regulus. But the two reguli that determine the tangent S_3's with G and the S_3's of G and the near by lines, are conjugate reguli of the same quadric. Thus, if G cuts one regulus it cuts the other also. Hence the theorem.

In line space this theorem becomes: *The tangent linear congruences to m at lines of a pencil g have in common a second pencil h having a line in common with g if, and only if, the linear congruences determined by g and the pencils of m infinitely near g have in common a second pencil h′ having a line in common with g.*

If all the S_3's determined by G, and the lines of M infinitely near G have in common a plane of Q, they all coincide. Thus, *if all the linear congruences determined by g and the pencils of m infinitely near g have in common anything besides a second pencil having a line in common with g, they coincide.*

If all the tangent S_3's to M along the generators of M have a plane in common, M is said to be developable of the first kind. We will also call developable of the first kind the corresponding m-complexes, viz., those whose tangent linear congruences along the pencils of m have in common besides g, a second pencil having a line in common with g.

6. The linear congruence determined by two pencils is that congruence which contains the two pencils, and therefore the congruence which has for directrices the line joining the centers of the two pencils and the line of intersection of the planes of the two pencils. Hence the directrices of a linear congruence determined by a pencil g and a second pencil of m infinitely near g are a tangent line to S at the center of g, and a tangent line to s at the point of contact of g.

Let us consider the nature of these surfaces S and s when m is developable of the first kind. That is all the linear congruences whose directrices are corresponding tangent lines to S and s have in common g and a second pencil h which has a line y in common with g. Let P be the center and π the plane of g. P lies on S and π is tangent to s at P'. The tangent plane to S at P is π'. Then the directrices of the linear congruences referred to above will be corresponding lines of the pencils $P\pi'$ and $\pi P'$. If all of these congruences are to contain h, every line of h must cut every line of the pencils $P\pi'$ and $\pi P'$. Since h is different from g it must then be the pencil $P'\pi'$; and y, the line joining corresponding points of S and s, is tangent to both these surfaces. Hence the lines y form a congruence whose focal surfaces are S and s.

Conversely, the two-parameter family of pencils having as centers one set of focal points of a line congruence and planes, the corresponding focal planes of the congruence is an m-complex of the first developable kind. Thus every line congruence generates two m-complexes of this kind.

7. Segre* has shown that if M is a V_3 in S_5 of the first developable kind the lines of M are all tangent to a surface or else all cut a curve. Let us consider first the case in which all the lines of M are tangent to a surface. This surface lies on Q and may be taken as the α-surface. At every point of α there are two lines tangent to α and lying on Q. β may be chosen as any surface cutting one set of these lines, for example,

$$\beta=\alpha_u+\rho\alpha_v. \tag{7}$$

But β must be a point on Q, hence

$$\rho^2(\alpha_u\alpha_u)+2\rho(\alpha_u\alpha_v)+(\alpha_v\alpha_v)=0. \tag{8}$$

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* Segre, *loc. cit.*, No. 29.

The two values of ρ obtained by solving (8) are the two directions of lines tangent to α and lying on Q. Call them ρ_1 and ρ_2. Let G_1 and G_2 be the two corresponding lines tangent to α. The locus of lines G_1 will be M_1 and the locus of G_2 will be M_2. The points of (4) determining the tangent S_3 become

$$(\alpha),\quad (\alpha_u+\rho\alpha_v),\quad [\alpha_{uu}+2\rho\alpha_{uv}+\rho^2\alpha_{vv}+\alpha_v(\rho_u+\rho\rho_v)\alpha_u+t(\alpha_{uu}+\rho\alpha_{uv})+t\rho_u\alpha_v], \tag{9}$$

and the points of (5) determining the S_3 of G and a line of M infinitely near G become

$$(\alpha),\quad (\alpha_u),\quad (\alpha_v),\quad [(\alpha_{uu}+\rho\alpha_{uv})\Delta u+(\alpha_{uv}+\rho\alpha_{vv})\Delta v]. \tag{10}$$

Thus, when M is a developable of the first sort the tangent S_3's to M along G, and the S_3's of G and the lines of M infinitely near G form two pencils of S_3's which lie in the same S_4 of (6). The plane axis of pencil (9) (as we vary t) depends on ρ, but that of (10) does not. Thus (9) furnishes two distinct pencils (9_1) and (9_2), while (10) furnishes two pencils (10_1) and (10_2) with the same central plane.

G_1 lies in two planes on Q, π_1 and π_1' and, similarly, G_2 in π_2 and π_2'. Since π_1 and π_2' are in different systems of planes of Q they intersect in a line L, and π_2 and π_1' intersect in L'. The four pencils of (9) and (10) cut on L and L' projective ranges. Projective ranges are also set up on G_1 and G_2 by means of (9_2) and (9_1), respectively.

In S_3 the congruence which corresponds to the tangent α-surface of S_5 is the y-congruence. To M_1 and M_2 there correspond the two m-complexes m_1 and m_2 mentioned in No. 6 as generated by the y-congruence. Thus, *if the m-complex is developable of the first sort and the y-congruence is not degenerate (i. e., a ruled surface) the tangent linear congruences to m along g form a pencil of congruences, and the linear congruences whose directrices are the corresponding tangent lines to the focal surfaces of the y-congruence also form a pencil of congruences. The centers (two pencils with a common line) of the last-named pencils of congruences for* m_1 *and* m_2 *coincide, while those of the first named do not.*

The lines L and L' correspond in S_3 to the pencils of tangent lines at corresponding points of S and s. Hence the above pencils of congruences cut off projective pencils of lines on these pencils.*

The necessary and sufficient condition that g and h coincide, *i. e.*, that S and s coincide is that the roots of (8) be equal. Hence

$$(\alpha_u\alpha_v)^2-(\alpha_u\alpha_u)(\alpha_v\alpha_v)=0\,\dagger \tag{11}$$

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* These are the projectivities of Waelsch, see "Zur Infinitesimalgeometrie der Strahlencongruenzen und Flächen," *Sitzungsbericht Akad. Wien Mathem. Classe*, i. 100 Abth. IIa (1891).

† Waelsch, *loc. cit.*, obtains the same result in a different way.

is a necessary and sufficient condition that S and s coincide. It will appear later that this is a necessary and sufficient condition that M have a fixed tangent S_3 along every generator.

Since pencils (10_1) and (10_2) have the same center, a necessary and sufficient condition that they cut off the same projectivities on L and L' is that they lie in the same S_4. A necessary and sufficient condition for this is

$$|\alpha, \quad \alpha_u, \quad \alpha_v, \quad \alpha_{uu}, \quad \alpha_{uv}, \quad \alpha_{vv}| = 0, \tag{12}$$

that is, the α-surface is a ϕ-surface of Segre.*

In line space this reads: A necessary and sufficient condition that the two pencils of linear congruences whose directrices are corresponding tangent lines of S and s should cut off the same projectivities on the pencils of lines tangent to S and s at corresponding points is that the y-congruence satisfy (12). But this says that the y-congruence is a W-congruence.†

A necessary and sufficient condition that the plane common to all the tangent S_3's to M along G should lie on Q, that is a necessary and sufficient condition that the tangent linear congruences to m at lines of a pencil g have in common a plane of lines or a bundle of lines, is that the plane of the first three points of (9) lie on Q. But this plane is the osculating plane to the curve of α, along which the lines of M are tangent. Hence the above condition becomes that the curves on α, along which the lines of M are tangent, should have their osculating planes lying on Q.

If α is a ruled surface the two M's defined are first, M_1 consists of the tangent lines to α along the rulings. Then M_1 is a two-dimensional variety, α, and not a three as we have considered. Second, M_2 consists of the other set of lines tangent to α and lying on Q. The corresponding y-congruence in S_3 is made up of ∞^1 pencils of lines, and the surfaces S and s are a curve and a developable, respectively. m_1 is a family of the sort described in Part I. If α is a developable it is a plane.

If α is cut by ∞^1 planes of Q in curves, M_1 will consist of the lines tangent to these plane curves. Conversely, if the lines of M can be grouped into the tangents to ∞^1 plane curves, these planes lie on Q. In the corresponding case in S_3 the y-congruence is made up of the lines of ∞^1 cones or the tangent lines to ∞^1 plane curves. Then either S will be a curve and s a non-specialized

* Segre, "Su una classe di superficie degli iperspazi legata colle equazione lineare alle derivate parziali di 2 ordine," *Atti della R. Accad. delle Scienze di Torino*, Vol. XLIX (1913–14), p. 215.

† Darboux, "Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal," Edition II (1889), p. 345.

surface, or s will be a developable and S a non-specialized surface. Conversely, if one of the surfaces is a curve or a developable, the surface α can be cut by ∞^1 planes of Q in curves. If both S and s are curves, or both are developables, the two sets of lines tangent to α and lying on Q will be tangent to α along plane curves.

If the lines of M, instead of being tangent to a surface, all cut a curve on Q, ∞^1 of them pass through each point of this curve. Then all the pencils of m have a line in common with a ruled surface. Hence, S and s are this surface, but S and s do not coincide in the sense that the center of the pencil of m on S is the point of contact of the pencil with s. The y-congruence also reduces to this surface.

8. It may happen in S_5 that the tangent S_3's to M along G coincide, that is, there is a fixed tangent S_3 to M along every generator. Then M is said to be developable of the second kind. If M is a developable of the second kind, the α and β surfaces of (3) satisfy two linear homogeneous partial differential equations as follows:

$$\left.\begin{array}{l} A_1\alpha_i+B_1\beta_i+C_1\alpha_{i_u}+D_1\beta_{i_u}+E_1\alpha_{i_v}+F_1\beta_{i_v}=0, \\ A_2\alpha_i+B_2\beta_i+C_2\alpha_{i_u}+D_2\beta_{i_u}+E_2\alpha_{i_v}+F_2\beta_{i_v}=0, \end{array}\right\} \; i=0,\,1,\,\ldots,\,5, \qquad (13)$$

where $A_j,\, B_j,\, \ldots,\, F_j$, $j=1,\,2$ are functions of u and v, and not all identically zero. Conversely, if α and β satisfy (13), M is a developable of the second sort. For under these circumstances the eight points of (6) all lie in an S_3, the tangent S_3 to M along G. Accordingly, if M is developable of the second kind, all the S_3's determined by G and the lines of M infinitely near G coincide. In fact, all the lines of M infinitely near G under these hypotheses, lie in the two planes of Q that contain G. For they all lie in the tangent S_3 to M along G which cuts Q in the two planes containing G.

On the other hand, if all the lines of M infinitely near G intersect G, M is a developable of the second kind. Thus, in S_3 if m is developable of the second kind, all the lines of pencils of m infinitely near g, lie in a special linear congruence with intersecting but distinct directrices which belong to the pencil g. Every pencil of m infinitely near g has a line in common with g and, conversely, if every pencil of m infinitely near g has a line in common with g the m-complex is developable of the second kind.

9. Segre has shown* that if M is developable of the second kind, all the lines of M are tangent to two surfaces of Q, or cut a curve of Q and are tangent to a surface, or cut two curves of Q. Since we know from No. 8 that if

* Segre, "Preliminari ....," No. 29.

m is developable of the second sort all the pencils of m infinitely near g have a line in common with g, we see that either they have their centers in the plane of g or their planes pass through the center of g. In either case S and s coincide. Hence, if m is developable of the second kind, S and s coincide. See (11). Conversely, if S and s coincide, m is developable of the second kind. By the theorem of Segre every pencil of m is tangent to two congruences (or tangent to a congruence, and has a line in common with a ruled surface, or has a line in common with two ruled surfaces). That is there exist two congruences, excluding for the moment the other two cases, such that every pencil of m has two infinitely near lines in common with each. Thus S must be the focal surfaces (coincident) for both of them, and they consist of the tangent lines to S along the two sets of asymptotic lines.

If the lines of M all cut a curve and are tangent to a surface of Q, the tangents to one set of asymptotics of S involve only ∞^1 lines and therefore S is a ruled surface. Similarly, if the lines of M all cut two curves, S has two sets of rulings and is therefore a quadric.

PART III.

10. A three-parameter family of lines on Q in S_5 may generate either a V_3 or a V_4. If they generate a V_3, that V_3 is either the intersection of Q with an S_4, or consists of ∞^1 planes imbedded on Q. We will demonstrate this in two parts. First let us assume that the V_3 is developable of the second kind. That is the tangent S_3 to V_3 along G is fixed. Through every point of V_3 there are ∞^1 lines of V_3, which lie in the tangent S_3 to V_3 at that point. But this S_3 is tangent to Q along G, and therefore cuts Q in the two planes that contain G. Hence the lines of V_3 in that S_3 lie in one or two planes of Q, and V_3 consists of ∞^1 planes of Q.

If V_3 is not developable of the second kind the tangent S_3's along G cut it in ∞^1 cones of lines which are distinct and V_3 is the locus of these cones. Through every point of V_3 there are ∞^1 lines and every line cuts ∞^2 others. Hence some line besides G in one of these cones cuts some other cone, and the two S_3's of these cones have in common besides G another point and therefore a plane. Hence all the tangent S_3's to V_3 along G lie in an S_4 according to No. 5. Hence V_3 is the intersection of Q with this S_4. Hence, in line space a three-parameter family of pencils of lines either involves all the lines of space or else ∞^3 of them. If they all lie in a complex, that complex is either a linear complex or else is made up of the lines in ∞^1 planes or through ∞^1 points.

11. In what follows let M_4 represent the locus of ∞^3 lines of Q which do not lie on a V_3. Then M_4 coincides with Q. Hence all the tangent S_4's to M_4 along a generator G have an S_3 in common. Therefore, according to Segre,* all the lines of M_4 are tangent to two V_3's, or either V_3 may be replaced by a director variety of less dimension.

The S_3's of G and the lines of M_4 infinitely near G are ∞^2 in number and may be grouped into ∞^1 pencils of S_3's whose axial planes themselves form a pencil of planes about G. A proof of this theorem follows M_4 can be represented analytically by means of

$$x_i = \alpha_i(u, v, w) + t\beta_i(u, v, w) \qquad i=0, 1, 2, \ldots, 5, \tag{14}$$

where α and β are the two tangent or director varieties mentioned in the theorem of Segre. Therefore

$$\left.\begin{aligned} \sigma\alpha &= \lambda\beta + \mu\beta_u + \nu\beta_v + \rho\beta_w, \\ \sigma\beta &= \lambda'\alpha + \mu'\alpha_u + \nu'\alpha_v + \rho'\alpha_w. \end{aligned}\right\} \tag{15}$$

The lines of M_4 infinitely near G are the lines from

$$(\alpha + \alpha_u\Delta u + \alpha_v\Delta v + \alpha_w\Delta w) \quad \text{to} \quad (\beta + \beta_u\Delta u + \beta_v\Delta v + \beta_w\Delta w),$$

where Δu, Δv, and Δw are infinitesimal. Hence the S_3's of G and the lines of M_4 infinitely near G are the S_3's of

$$(\alpha), \quad (\beta), \quad (\alpha_u\Delta u + \alpha_v\Delta v + \alpha_w\Delta w), \quad (\beta_u\Delta u + \beta_v\Delta v + \beta_w\Delta w). \tag{16}$$

As we vary the ratios $\Delta u : \Delta v : \Delta w$ in (16) we get the ∞^2 S_3's determined by G and the lines of M_4 infinitely near G. Call the plane of (α_u), (α_v), and (α_w) the plane π_α and similarly the plane π_β is the plane of (β_u), (β_v), and (β_w). According to (15) G cuts π_α in P_α and cuts π_β in P_β. Points represented by the same ratios of Δu, Δv, and Δw are corresponding points in the planes π_α and π_β. A line in π_α corresponds to a line in π_β. Let H be a line in π_α through P_α, and H' the corresponding line in π_β. Let K be a line joining corresponding points of H and H'. Then the S_3 of G and K is one of the S_3's of (16), and it contains the plane of G and H. As K moves along H these S_3's form a pencil of S_3's about the plane GH. But H was any line of π_α through P_α. Hence these planes themselves form a pencil about the line G. Hence the theorem. Similarly the S_3's of G and the lines of M_4 infinitely near G can be grouped into ∞^1 pencils of S_3's whose axial planes are determined by G and lines in π_β through P_β, that is, the axial planes form a pencil of planes about G in the tangent S_3's to π_α and π_β.

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* Segre, "Preliminari ....," No. 29.

In line space the preceding facts give us that: All the tangent linear complexes to m_4 at lines of a pencil g are special, with the lines of contact of the pencil g as axes, and have in common a linear congruence whose directrices belong to the pencil g. The linear congruences determined by g and the pencils of m_4 infinitely near g can be grouped into ∞^1 pencils of congruences having in common besides g a second pencil h which has a line in common with g. The locus of these pairs of pencils is a special linear congruence whose directrices are coincident. This last congruence is tangent to α or β at the line which g has in common with α or β, where α and β are line complexes to which all the pencils of m_4 are tangent (or congruences or ruled surfaces with which every pencil of m_4 has a line in common). The m_4 may be considered as the locus of ∞^1 m's of the first developable kind as in Part II.

The ordering of M_4 into ∞^1 m's of the first developable kind can be accomplished in only two ways corresponding to the two ways of ordering the S_3's of G, and the lines of M_4 infinitely near G into ∞^1 pencils of S_3's. For let $f(u, v, w)=0$ be any relation on the parameters u, v, w which yields a developable V_3 of the first kind on M_4. Then

$$f_u\Delta u+f_v\Delta v+f_w\Delta w=0, \tag{17}$$

and (17) represents a line H in π_α, and a corresponding line H' in π_β. Lines joining corresponding points of H and H' form a regulus, and the S_3's of G and the lines of this regulus are the S_3's of G and the lines of M_4 infinitely near G, and lying on the V_3, determined by $f=0$. If this V_3 is to be developable of the first kind all these S_3's must have a plane in common and G must cut this regulus. Let us assume that this intersection takes place outside of π_α and π_β. Then the line of the regulus which cuts G cuts π_α in Q_α and π_β in Q_β. Hence P_α, P_β, Q_α, and Q_β are coplanar, and π_α, π_β, and G all lie in the same S_4. Hence the points (α), (β), (α_u), (α_v), (α_w), (β_u), (β_v), (β_w) all lie in an S_4, and the tangent S_4 to M_4, and therefore to Q along G is fixed, which is impossible. Hence the assumption that G cuts the above regulus in a point outside of π_α and π_β is false. Hence H passes through P_α or H' passes through P_β and the two cases of the preceding paragraph are unique.

Hence the only f's which give developable V_3's of the first kind are those for which the line (17) passes through P_α or P_β. The coordinates of these points may be obtained from (15), the point $(\sigma\alpha-\lambda\beta)$ is the point of G in π_β, and therefore has for coordinates $\mu:\nu:\rho$ which is the point P_β, similarly P_α is

$\mu' : \nu' : \rho'$. Hence the developables of the first kind on M_4 and on m_4 are given by one of the following equations:

$$\left.\begin{aligned} \mu f_u + \nu f_v + \rho f_w &= 0, \\ \mu' f_u + \nu' f_v + \rho' f_w &= 0. \end{aligned}\right\} \quad (18)$$

If α or β or both are surfaces instead of V_3's, there will exist a relation on the three parameters of that variety and one of them can be eliminated. Hence either μ, ν or ρ is zero, or one of μ', ν', ρ' is zero, or both. This will not affect the above reasoning. If either α or β is a curve, the pencils of S_3's determined by G and the lines of the regulus from H to H' collapse into single S_3's, for then either H or H' is a point. That is, the M_4 can be grouped into ∞^1 M's of the second developable kind.

Thus M_4's may be classified according to the type of differential equations (15) that they satisfy. If all the coefficients μ, ν, ρ and μ', ν', ρ' are different from zero, M_4 is made up of lines tangent to two V_3's. If one of these coefficients is zero, the lines of M_4 are tangent to a V_3 and cut a surface. If one of the primed and one of the unprimed coefficients are zero, the lines of M_4 cut two surfaces. If two of the primed or two of the unprimed coefficients are zero, the lines of M_4 are tangent to a V_3 and cut a curve. Other cases can not occur.

Similarly m_4's may be grouped according to the type of differential equations (15) the α and β complexes satisfy. If none of these coefficients is zero, the pencils of m_4 are tangent to two complexes. If one is zero they are tangent to a complex and have a line in common with a congruence. If two are zero in different equations they have a line in common with two congruences. If two are zero in the same equation they are tangent to a complex and have a line in common with a ruled surface. In the last case the pencils can be grouped into ∞^1 m's of the second developable type.

12. If the lines of M_4 can be grouped into ∞^1 M's of the second developable type, there must exist an $f(u, v, w) = 0$ for which the tangent S_3 along G is fixed, and therefore, lies in both the tangent S_4 at α, viz., $(\alpha x) = 0$, and the tangent S_4 at β, viz., $(\beta x) = 0$. Hence the last two points of (16) must lie in the S_3 of intersection of $(\alpha x) = 0$ with $(\beta x) = 0$ for all values of $\Delta u : \Delta v : \Delta w$ which satisfy (17). These four conditions reduce to

$$(\alpha\beta_u)\Delta u + (\alpha\beta_v)\Delta v + (\alpha\beta_w)\Delta w = 0 \text{ by virtue of (17).}$$

Hence

$$(\alpha\beta_u) : (\alpha\beta_v) : (\alpha\beta_w) = f_u : f_v : f_w. \quad (19)$$

And (19) is a necessary and sufficient condition that the lines of M_4 be susceptible to an arrangement into ∞^1 M's of the second developable kind. Furthermore (19) gives us the f's that are developable.

Hence (19) is a necessary and sufficient condition that the pencils of m_4 be capable of grouping into ∞^1 m's of the second developable type, that is whose focal surfaces, S and s, coincide, or such that the locus of the centers of the pencils and the envelope of their planes coincide.

The Pfaffian differential equation

$$X(x, y, z)\,dx + Y(x, y, z)\,dy + Z(x, y, z)\,dz = 0 \qquad (20)$$

defines precisely a three-parameter family of pencils of lines in ordinary space. (19) is the condition of integrability. The preceding work shows how to pick out in Klein coordinates the two-parameter families of pencils of lines for which the centers of the pencils lie on a surface S, and their planes are tangent to a surface s, where S and s are of such a nature that lines joining corresponding points of each are tangent to both in the non-integrable case. Voss * has discussed three-parameter families of pencils from this point of view.

UNIVERSITY OF MICHIGAN, Ann Arbor, Mich., *March* 17, 1917.

* Voss, "Zur Theorie den allgemeinen Punktebenesysteme," *Math. Annalen*, Vol. XXIII, p. 45.

Some Contributions to the Geometry of Plane Transformations.

By Tobias Dantzig.

§ 1. *The Indicatrix of a Transformation.*

Let T be a continuous plane point-to-point transformation sending a plane Π into itself. To fix the idea I shall further assume that T transforms every point P in the plane Π into a unique point $\overline{P}$. The latter I shall call the image of P by T. In the concluding pages of this paper is to be found an extension of most of the considerations here developed to the case of a $p:q$ transformation.

Let P be a point of the plane Π and let $\overline{P}$ be its image by T. Consider an arbitrary curve C passing through P and let t be its tangent at the point P. The curve C is transformed by T into a curve $\overline{C}$ which will pass through $\overline{P}$ and have there a unique tangent $\bar{t}$, providing the point P is a simple point of the curve C. Let the two lines t and $\bar{t}$ meet in a point τ. Then it is easy to see that,

The position of the point τ is independent of the curve C and is perfectly determined once the points P and $\overline{P}$ and the line t are given.

Indeed let C' be any other curve passing through P and having there t for tangent. The image $\overline{C}'$ will pass through $\overline{P}$ and, *since T conserves contact*, will touch at $\overline{P}$ the line $\bar{t}$. The point τ I shall call the *tactal* of the line t at the point P.

Assuming now the points P and $\overline{P}$ fixed, to each line t through P will correspond a unique line $\bar{t}$ through $\overline{P}$. When the line t turns about P a *one-to-one correspondence* is established between the two pencils having their vertices in P and $\overline{P}$, respectively, and the point is the *product* of two corresponding rays. It follows at once, therefore, that

The locus of the tactal τ for one and the same point P, is a conic K passing through P and $\overline{P}$.

The conic K is uniquely determined for every point P of the plane Π. I shall call it the *indicatrix of the transformation T for the point P.* The totality of all the indicatrices of the plane form a system of conics depending on two parameters.

When the points P and $\overline{P}$ are known, the determination of K requires the knowledge of *three* tactals. These can be obtained by tracing through P three arbitrary curves having different tangents at P. The proper selection of these curves will facilitate the construction of K. The following obvious remarks may help to guide in this selection.

The line $P\overline{P}$ I shall call the *bridge* of the transformation at P. It follows at once that

If a curve C touches the bridge $P\overline{P}$ at P, its image $\overline{C}$ will touch at $\overline{P}$ the indicatrix K, and, conversely, if C touches K at P, $\overline{C}$ will touch $P\overline{P}$ at $\overline{P}$.

If C and $\overline{C}$ have parallel tangents at corresponding points P and $\overline{P}$, the direction of these tangents is an asymptotic direction of the indicatrix K for P.

The importance of the introduction of the indicatrix can be best illustrated by the following problem:

Determine a geometrical construction of the tangent $\overline{t}$ at a point $\overline{P}$ of a plane curve $\overline{C}$, knowing that $\overline{C}$ is the image of a curve C by a transformation T.

Assume that the geometrical construction of the tangent t of the curve C is known, and let K be the indicatrix of T for the corresponding point P. If t meets K in a second point τ, τ is evidently the tactal of t at P, and consequently $\tau\overline{P}$ is the tangent sought. It is interesting to remark that if K is determined by five points (P, $\overline{P}$ and three tactals τ_1, τ_2, τ_3) and t is known, the *construction can be completed by projective means only* (Pascal's theorem).

These considerations bring out the following property: The indicatrix of a transformation T for a point P can serve as a *first asymptotic element of T at P, characterizing the behavior of T in the vicinity of the point P,* to the same degree as the tangent at a point of a plane curve characterizes the behavior of the curve at the point.

§ 2. *Equation of Indicatrix.*

To obtain the general equation of the indicatrix assume that the equations of the transformations T are given in the explicit form

$$\overline{x}=\phi(x, y), \quad \overline{y}=\psi(x, y), \tag{1}$$

where ϕ and ψ are single-valued continuous functions of x and y, admitting first derivatives which are also single valued and continuous; and where x, y;

$\bar{x}$, $\bar{y}$ are the coordinates of P and $\bar{P}$, respectively, in a system of cartesian coordinates. We shall set

$$\frac{\partial\phi}{\partial x}=p, \quad \frac{\partial\phi}{\partial y}=q, \quad \frac{\partial\psi}{\partial x}=p', \quad \frac{\partial\psi}{\partial y}=q'.$$

The equations of the corresponding tangents t and $\bar{t}$ at P and $\bar{P}$ being evidently

$$\frac{X-x}{dx}=\frac{Y-y}{dy}, \text{ and } \frac{X-\bar{x}}{d\bar{x}}=\frac{Y-\bar{y}}{d\bar{y}}, \tag{2}$$

we are to eliminate the differentials from these equations with the aid of the identical relations

$$d\bar{x}=pdx+qdy, \quad d\bar{y}=p'dx+q'dy. \tag{3}$$

We have, by setting the common ratios in (2) equal to $1/\rho$ and $1/\bar{\rho}$, respectively, the two equations

$$\bar{\rho}(X-\bar{x})=[p(X-x)+q(Y-y)]\rho, \quad \bar{\rho}(Y-\bar{y})=[p'(X-x)+q(Y-y)]\rho,$$

and eliminating ρ and $\bar{\rho}$ we obtain the equation of the indicatrix in the form

$$\frac{p(X-x)+q(Y-y)}{X-\bar{x}}=\frac{p'(X-x)+q'(Y-y)}{Y-\bar{y}}. \tag{4}$$

In this form it is apparent that the indicatrix is circumscribed to the quadrilateral whose four sides are $X=\bar{x}$, $Y=\bar{y}$, $p(X-x)+q(Y-y)=0$, $p'(X-x)+q'(Y-y)=0$. Collecting the terms we have

$$p'(X-x)(X-\bar{x})-p(X-x)(Y-\bar{y})-q'(X-\bar{x})(Y-y)$$
$$-q(Y-y)(Y-\bar{y})=0. \tag{5}$$

The character of the conic K is determined by the discriminant of the quadratic part of equation (5)

$$\Delta(x, y)=(p-q')^2+4p'q. \tag{6}$$

Assuming now that the transformation T is real, the curve $\Delta(x, y)=0$ divides the plane Π into two *regions: E* and *H*. In the region E, $\Delta<0$ and the indicatrix K is an *ellipse*, in the region H, $\Delta>0$ and the conic K is a *hyperbola*, on the *discriminant curve* itself the conic is a parabola. I shall call these regions the *elliptic* and *hyperbolic regions* respectively. If the transformation admits the whole plane for an *elliptic region* I shall call it an *elliptic transformation*, and the terms *hyperbolic* and *parabolic* transformations should be taken in the same sense.

From the consideration of the preceding section it follows at once that

At any point P in the region H there exist two rays α and β, such that, if a curve C passing through P admits one of the rays for tangent, its image $\overline{C}$ will have a parallel tangent at $\overline{P}$. These rays α and β are the asymptotic directions of the indicatrix K. They are coincident at every point of the discriminant curve Δ and become imaginary in the region E.

Of course, the locus $\Delta=0$ is not necessarily *singly connected.* Consider for instance the *transcendental* transformation

$$\bar{x}=\cos y, \quad \bar{y}=\sin x.$$

We have here

$$\Delta=-4\cos x \cdot \sin y,$$

and the discriminant locus consists of the lines

$$x=\pi/2+k\pi \quad \text{and} \quad y=l\pi.$$

The locus divides the plane into an *infinite number of squares* whose sides are parallel to the coordinate axes and which alternately constitute the *elliptic* and *hyperbolic regions.*

§ 3. *The Invariant Points.*

The *invariant* points of a transformation T given by equations (1) are obtained by setting

$$\bar{x}-x=\xi=0, \quad \bar{y}-y=\eta=0. \tag{7}$$

If equations (7) admit but a discrete number of simultaneous solutions, the invariant points are isolated. It may happen, however, that the equations $\xi=0$, $\eta=0$ have a factor $f(x, y)$ in common. The curve $f=0$ is then a *curve of fixed points.*

I shall prove now that:

If S is an invariant point the indicatrix K for S degenerates into two straight lines passing through S.

Indeed, the image $\overline{C}$ of an arbitrary curve C, passing through S must also go through S. If then the tangents t and $\bar{t}$ are distinct, the tactal of t is in S, if they coincide, the tactal is indetermined on t. The coincidence of t and t will occur only when the tangent is an asymptotic direction of K. Consequently the *conic K will degenerate at an invariant point into its asymptotes.*

According as the invariant point is situated in the elliptic or hyperbolic regions, or on the discriminant curve, we shall call the invariant points *elliptic, hyperbolic, or parabolic.*

An elliptic invariant point is necessarily isolated, if real.

Indeed the indicatrix degenerates into two imaginary lines at such a point. If there were another real invariant point S', in the vicinity of S, and at an infinitesimal distance from it, SS' would necessarily be an asymptotic direction and the latter would be real, contrary to the hypothesis.

As a corollary we see at once *that in the region E no real curve can touch its image at an invariant point.*

If then T admits of a curve of fixed points the latter must be entirely in the region H or make part of the discriminant curve $\Delta=0$. In either case the curve of fixed points is the envelope of one of the branches of the indicatrix K.

Conversely, if the indicatrix of a point P degenerates into two lines passing through P, the point P is necessarily an invariant point.

EXAMPLE: Consider the transformation defined by the equations

$$\bar{x}=(x^2+y^2+2ax-a^2)/2a, \quad \bar{y}=(x^2+y^2+2ay-a^2)/2a.$$

We have in this case

$$\Delta=[(x+y)/a]^2.$$

The discriminant curve is the line $x+y=0$ counted double. The transformation is hyperbolic throughout the plane, except on Δ, where the indicatrix is a parabola. We have here

$$\xi=\eta=(x^2+y^2-a^2)/2a,$$

and the transformation admits of a fixed circle

$$x^2+y^2=a^2.$$

At any point of this circle the indicatrix degenerates into the tangent to the circle and the line parallel to $x=y$.

§ 4. *The Jacobian.*

The degeneracy of the indicatrix at an invariant point leads us to examine the general case of degeneracy of the conic K. For this purpose make in (5) the substitution

$$\xi=\bar{x}-x, \quad \eta=\bar{y}-y.$$

The equation of K then becomes

$$p'(X-x)^2-(p-q')(X-x)(Y-y)-q(Y-y)^2 \\ -(p'\xi-p\eta)(X-x)-(q'\xi-q\eta)(Y-y)=0. \quad (8)$$

The condition that the conic degenerates is

$$(pq'-qp')[p'\xi^2-(p-q')\xi\eta-q\eta^2]=0. \quad (9)$$

The degeneracy locus therefore consists of the two curves

$$J(x, y) = pq' - qp' = 0, \tag{10}$$

and

$$D(x, y) = p'\xi^2 - (p - q')\xi\eta - q\eta^2 = 0. \tag{11}$$

The first is the *Jacobian of the transformation*, the second I shall call *the degeneracy locus proper*. We see that the invariant points make part of the locus $D=0$.

Geometrically the two cases can be characterized in the following manner. If P is not an invariant point it may happen that the degenerate conic has one branch α passing through P, the other β passing through $\overline{P}$. The tactal of any line t is evidently on β; β is therefore the line corresponding to any direction t through P. The transformation therefore establishes a pseudo-correspondence between the two pencils. Such points are characterized by the fact that the image of an arbitrary curve passing through them has a fixed tangent β, independent of the direction of the tangent t. If m and $\overline{m}$ are the slopes of two curves C and C at corresponding points we have in general

$$\overline{m} = \frac{d\overline{y}}{d\overline{x}} = \frac{p' + q'm}{p + qm}, \tag{12}$$

and the condition that this homographic correspondence degenerates into a pseudo-correspondence is

$$J = 0.$$

At any point of the Jacobian of the transformation T the indicatrix degenerates into two lines of which but one α passes through P, the other β containing the point $\overline{P}$. Conversely, if the degeneracy is of this type, the point P is on the Jacobian.

It follows from these considerations that *if a curve C has in a point P belonging to the Jacobian of T a double point, its image $\overline{C}$ will have in $\overline{P}$ a cusp, and the cuspidal tangent is the branch β of the indicatrix.*

§ 5. *The Degeneracy Locus Proper.*

We shall now interpret the equation (11)

$$D = p'\xi^2 - (\gamma - q')\xi\eta - q\eta^2 = 0.$$

The equation of the indicatrix in this case is

$$p'(X-x)^2 - (\gamma - q')(X-x)(Y-y) - q(Y-y)^2 = 0, \tag{13}$$

and the condition $D=0$ expresses that one branch of the degenerate conic coincides with the bridge

$$(X-x)/\xi = (Y-y)/\eta. \tag{14}$$

At every point of the locus $D=0$ the indicatrix degenerates into two lines of which one α coincides with the bridge at P. *If the bridge is a tangent to any curve C passing through P, it is also tangent to the transformed curve C at P. Conversely, if the degeneracy is of this type, the point P at which it occurs belongs to the locus $D=0$.*

If we set $\omega=\eta/\xi$ and take in consideration that

$$\frac{\partial\xi}{\partial x}=p-1,\quad \frac{\partial\xi}{\partial y}=q,\quad \frac{\partial\eta}{\partial x}=p',\quad \frac{\partial\eta}{\partial y}=q'-1,$$

equation (11) takes the remarkable form

$$\frac{\partial\omega}{\partial x}+\omega\frac{\partial\omega}{\partial y}=0. \tag{15}$$

Consider the curves, $\omega(x, y)=\text{const.}$ Through any point in the plane passes a unique curve of this family. The transformation T *shifts every point of the curve, $\omega=const.$, in a constant direction whose slope is ω.* The slope of the tangent at any point of the curve is equal to

$$m=-\frac{\partial\omega}{\partial x}\Big/\frac{\partial\omega}{\partial y}.$$

Relation (15) expresses that at a point where any of the curves $\omega=\text{const.}$ crosses the degeneracy locus D *the curve touches the bridge of the point.*

Consider in particular a line l which the transformation T leaves invariant. At every point P of such a line the indicatrix degenerates into the line l itself and a second line which in general crosses l in a point different from P or $\overline{P}$. It follows therefore that

An invariant straight line of a transformation necessarily makes part of the degeneracy locus $D=0$. At all points of such a line the indicatrix degenerates and one branch of it is the line l itself.

From the discussion of Section 2 we conclude that *a real invariant line can not penetrate into the elliptic region of the plane*, for the degenerate indicatrix is necessarily *real* at all points of such a line.

The same remark holds for any continuous branch of the degeneracy locus and the Jacobian: *they can have in the elliptic region of the plane but isolated points.*

As a corollary it follows that an elliptic transformation admits of no invariant lines unless *a part of its discriminant locus be an invariant straight line.*

§ 6. *Directed Transformations.*

The case when the indicatrix degenerates at any point in the plane deserves special attention. Equation (9) shows at once that this will happen: *First,* when we have identically

$$\xi=0, \quad \eta=0.$$

In this case we have $x=x$, $y=y$, and T is the *identical transformation. Second,* when J is identically zero. There exists then a functional relation between x and y, and T is a *pseudo-transformation.* Equation (1) is but a *parametrical representation* of a curve, or we can regard T as transforming the plane Π into a *one-dimensional configuration.* Leaving these cases which do not present any interest, we still have to interpret the identical vanishing of D. The general integral of equation (15) is easily found to be

$$y-\omega x=F(\omega), \tag{16}$$

where F is an arbitrary function and ω is the slope of the bridge of any point. Since, on the other hand, the equation of the bridge at a point $P(x, y)$ is

$$Y-\omega X=y-\omega x;$$

equation (16) expresses that the totality of the bridges in the plane depend on but one parameter. The bridges therefore envelope a curve W. We shall call this type of transformation a *directed transformation,* and the curve W *the directrix of T.*

If the directrix is an algebraic curve of *class k,* the multiplicity of such a transformation is at least k to k. The determination of such a transformation therefore requires the knowledge of the directrix and the *mode of correspondence* between the points P and $\overline{P}$ on the bridge. If the latter is a p-to-q correspondence, the multiplicity of the transformation is evidently $kp:kq$.

Assume, in particular, that the correspondence between the points on the bridge is *one-to-one.* All the *singular* points of the transformation are evidently situated either on the directrix or on one of the double tangents to the directrix. The transformation is *regular* at any other point of the plane, and any one of its branches may be regarded as a *one-to-one* transformation. *At any such regular point the indicatrix degenerates into the bridge and a second line β.**

* Concerning the meaning of singular and regular points see Section 21.

§ 7. *Central Transformation.*

If on a directed transformation we impose the additional condition of being of *multiplicity* 1:1, the directrix W must reduce to a *point* 0. The transformation is characterized by the fact that all bridges concur in a point, the center of the transformation.

The indicatrix of such a *central* transformation degenerates at any point in the plane into the ray of the point and a second line β.

From the preceding section it is seen that *the center is a singular point.* If we take it for origin, the equation of any central transformation can be put in the form

$$\bar{x}=g(x, y) \cdot x, \quad \bar{y}=g(x, y) \cdot y. \tag{17}$$

The curves $g(x, y)=$const. are of special interest. We have indeed,

$$\xi=(g-1) \cdot x, \quad \eta=(g-1) \cdot y, \tag{18}$$

and the locus $g=1$ is a curve of fixed points.* We find, on the other hand,

$$J=g(xg_x+yg_y+g),$$

and the *Jacobian* consists of the curves $g=0$ and $xg_x+yg_y+g=0$.

We find for the discriminant

$$\Delta=(xg_x+yg_y)^2, \tag{19}$$

and the two branches have for equation

$$yX-xY=0, \quad g_xX+g_yY=g_xx+g_yy-g(g-1). \tag{20}$$

Consequently *the second branch of the indicatrix is parallel to the tangent of the curve $g=const.$ passing through the point.* At any point of the discriminant curve the curve $g=$const. touches at P the radius OP. From this it follows that *the condition that a central transformation be parabolic is that the family of curves* $g=$const. *consists of the pencil of lines through* O. Analytically the condition can be obtained by integrating (19), which gives

$$\bar{x}=g(y/x) \cdot x, \quad \bar{y}=g(y/x) \cdot y. \tag{21}$$

If we put this in polar coordinates we arrive at the conclusion that the transformation

$$\bar{\Theta}=\Theta, \quad \bar{\rho}=g(\theta)\rho \tag{21'}$$

is *parabolic throughout the plane.*

The curve $g(x, y)=k$ and its image are evidently *homothetic* with respect to the origin, the *ratio of homothety* being k.

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* It is assumed here that if the function $g(x, y)$ has a denominator, this latter does not contain x or y as a factor.

§ 8. *The Foci of a Transformation.*

Whenever a transformation admits of an *invariant pencil* of straight lines I shall call the vertex of such a pencil *a focus* of the transformation. The following theorem brings out the importance of these points.

All indicatrices of a transformation pass through its foci, and conversely, if all the indicatrices of a transformation pass through a fixed point, the pencil of lines through this point is left invariant.

For let K be the indicatrix for any point P, $\overline{P}$ the image of P, and C a focus of the transformation. By hypothesis PC is transformed into $\overline{P}C$, C is therefore a tactal. Conversely, if all the indicatrices pass through a point C, consider any line l through C and the family of indicatrices relative to the points on l. To each point P corresponds on the conic K a point $\overline{P}$ such that $\overline{P}C$ is the tangent at $\overline{P}$ to the image of l. All the tangents of $\overline{l}$ concur therefore in C, which proves that $\overline{l}$ is a straight line passing through C.

The transformation being assumed one-to-one, there exists a *homographic correspondence* between the two *superposed pencils* l and $\overline{l}$. We see therefore, that in any invariant pencil there are *two invariant lines* u and v, the *double rays* of the pencil through C. I shall call the invariant pencil *elliptic, hyperbolic*, or *parabolic*, according as u and v are congugate imaginary, real, or coincident.

If an invariant pencil contains more than two invariant lines, every line in the pencil is invariant. The homographic correspondence between the two pencils l and $\overline{l}$ reduces to *identity* and the transformation is a *central* transformation.

I shall say that a transformation is *unifocal, bifocal*, etc., if it admits of one, two, etc., foci. It follows from the considerations developed that *among the indicatrices of a unifocal transformation there are two families of degenerate conics.* All the indicatrices of any one of the two families have an invariant line of the transformation for one common branch.

No one-to-one transformation can possess more than three non-collinear foci.

For, assume that there are four foci, C_1, C_2, C_3, C_4, no three of which are on a straight line. All indicatrices will have to pass through the four foci. The system of indicatrices in the plane would reduce to a *pencil of conics* and any one of the conics would have to serve as indicatrix for any point on it, which is absurd.

We shall see later that if a transformation admits of *three non-collinear foci* it is a *collineation.*

If three of the foci of a transformation are collinear, any point on the line δ containing these foci is also a focus.

Indeed, since any indicatrix must pass through all the foci, they will all have the *axis* δ for common branch, and, consequently, contain any point on δ. We shall see later that the transformation is a *perspective.*

§ 9. *Unifocal Transformation.*

If the focus of a unifocal transformation be taken for origin, one of the equations of the transformation can be readily obtained by expressing that there exists between y/x and $\bar{y}/\bar{x}$ a *homographic relation*

$$Ax\bar{y}+By\bar{y}=Cx\bar{x}+Dy\bar{x}. \tag{22}$$

By setting

$$\bar{x}/(Ax+By)=\bar{y}/(Cx+Dy)=g(x,y),$$

the equations of the transformation are put into the form

$$\bar{x}=(Ax+By)\cdot g(x,y),\quad \bar{y}=(Cx+Dy)\cdot g(x,y); \tag{23}$$

and this is the resultant of the two transformations

$$\bar{x}=x'g(x',y'),\quad \bar{y}=y'g(x',y')\quad \text{and}\quad x'=Ax+By,\quad y'=Cx+Dy.$$

Consequently,

Any unifocal one-to-one transformation is the product of a central and a linear transformation.

Let now T_1 and T_2 be two one-to-one transformations, one sending a point P into P_1, the other sending P_1 into P_2, and let $T_1T_2=T_3$ represent the product of the two transformations. Let, moreover, K_1 be the indicatrix of T_1 for P, K_2 that of T_2 for P_1, and K_3 that of T_3 for P. *The conic K_3 must go through the three intersections of K_1 and K_2 others than P_1.* Indeed, if Q be one of these intersections, QP and QP_1 are corresponding rays in the pencils (P) and (P_1). Since, on the other hand, P_2 is on K_2, QP_1 and QP_2 are corresponding rays in the pencils (P_1) and (P_2). It follows therefore that QP and QP_2 must be corresponding rays in the pencils (P) and (P_2) as determined by the transformation T_3. Q is therefore on the conic K_3.

This lemma permits the determination of the indicatrix of a product of two transformations when the indicatrices of the components are known. For it gives five points of the conic K_3: P, P_2, Q, Q_1, Q_2. In particular it can be applied to any unifocal transformation thus *reducing the determination of the indicatrix to that of a central transformation.*

§ 10. *Bifocal Transformation.*

Consider a one-to-one transformation T admitting *two* foci, C_1 and C_2. We shall call the line $l=C_1C_2$ the *line of foci.* Each of the pencils C contains *two invariant lines.* Denote these by u_1, v_1 and u_2, v_2, respectively. It is clear that the transformation admits of *four invariant points,* u_1u_2, u_1v_2, v_1u_2, v_1v_2. The foci are singular points; the image of C_1 is indetermined on the line l_2 corresponding to l in the pencil (C_2), and similarly C_2 is transformed into the line l_1 corresponding to l in the first pencil. The transformation admits of a third singular point G, the intersection of the lines l_1' and l_2' to which l corresponds in the pencils (C_1) and (C_2), respectively. The image of G is therefore the line of foci.

On the other hand T can be considered in two ways as a *unifocal transformation.* It follows that the equations of the transformation are of the form

$$A_1\bar{x}+B_1\bar{y}+C_1=0, \quad A_2\bar{x}+B_2\bar{y}+C_2=0, \tag{24}$$

where the coefficients are linear expressions in x and y.

T is therefore a "Magnus transformation."* Any straight line is transformed into a conic going through *three fundamental points,* which in this case are C_1, C_2 and the intersection F of l_1' and l_2'. Conversely, any conic through C_1, C_2 and the intersection G of l_1' and l_2' is transformed into a straight line. In particular, the pencil of lines through G goes into the pencil through F.

Let L, M, N be the fundamental points of any Magnus transformation T, and L', M', N' the fundamental points of the *inverse* transformation T^{-1}, and consider T as operating between two superposed planes Π and Π'. Then a linear transformation could bring two pairs of corresponding points in coincidence, say L' into L and M' into M. Under these circumstances the pencils (L') and (M') would be superposed with (L) and (M), and the points L and M would be foci of the new transformation. Therefore

The most general Magnus transformation can be considered as the product of a linear transformation and a bifocal transformation.

§ 11. *Conformal Transformations, Elliptic Type.*

If a transformation conserves angles both in size and in sense of rotation I shall call it an *elliptic* transformation, otherwise a *hyperbolic conformal transformation.*

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* See Magnus, "Nouvelle méthode," etc. *Crelle,* 8 pp. 52–63.

An elliptic conformal transformation has for indicatrix at any point of the plane a circle.

The theorem is evident geometrically. Indeed, let t and t' be any two rays of the pencil (P), and let $\bar{t}$ and $\bar{t}'$ be the corresponding two rays in the pencil $(\bar{P})$. By hypothesis the angles (t, t') and $(\bar{t}, \bar{t}')$ are equal and have the same sense of rotation; the locus of the point τ is therefore a circle.

Conversely, if the indicatrix is a circle, the angles are conserved. Analytically the condition that equation (8) represent a circle is given by

$$p=q', \quad q=-p', \tag{25}$$

and this is the well-known condition that a transformation be "*directly*" *conformal.*

A conformal transformation of this type is necessarily *elliptic* throughout the plane, and the discriminant curve is imaginary. *An invariant point of an elliptic conformal transformation is necessarily isolated and admits for indicatrix the isotropic lines passing through it.*

There are no real invariant lines.

It follows from the developments of the preceding section that *a conformal one-to-one transformation of this type is a bifocal transformation having two elliptic foci in the cyclic points at infinity I and J.*

All straight lines in the plane are transformed into conics going through the foci, *i. e.*, into *circles*, and these circles will have a third fixed point in common, real or imaginary. *The lines of the plane have for images circles having a radical center in common.* This radical center is *real*, for the lines l and l, to which the line at ∞ corresponds in the pencils I and J, respectively, are conjugate imaginary. For the same reason the *four invariant points of a conformal transformation are real.*

Let F be the radical center of the circles, and let G be the radical center of the circles which are transformed into lines. Then from the preceding section it is seen that F is transformed into the line at infinity and G is the image of the same line, *i. e.*, F and G are *vanishing points.*

§ 12. *Conformal Transformations, Hyperbolic Type.*

The case of the *hyperbolic conformal* transformation, in which the sense of rotation of angles is reversed, is connected with the preceding case by the following theorem:

Any hyperbolic conformal transformation is the product of a circular transformation and reflexion.

Indeed, let

$$\bar{x}=\phi(x, y), \quad \bar{y}=\psi(x, y)$$

be the equations of a circular transformation. If we make the substitution

$$x=x', \quad y=-y',$$

we will have

$$\frac{\partial \bar{x}}{\partial x'}=p, \quad \frac{\partial \bar{x}}{\partial y'}=-q, \quad \frac{\partial \bar{y}}{\partial x'}=p', \quad \frac{\partial \bar{y}}{\partial y'}=-q',$$

the condition $p=q'$, $p'=-q$ is therefore transformed into

$$p=-q', \quad p'=-q. \tag{26}$$

Nevertheless there are some advantages in treating the problem directly. The following theorem is true:

The indicatrix of a conformal transformation of the hyperbolic type is an equilateral hyperbola admitting the bridge for diameter.

The proof can be based on this classical property of an equilateral hyperbola: Any arc of the curve is viewed at equal or supplementary angles from the extremities of any diameter of the curve. It can also be regarded as the geometrical interpretation of conditions (26) in the equation of the indicatrix (8).

Let us apply these considerations to the determination of all *central conformal transformations.* It is clear that, if real, these can not be of the circular type as the indicatrix degenerates. Therefore the indicatrix is a degenerate equilateral hyperbola admitting the bridge for diameter. The second branch of the indicatrix is then the *perpendicular bisector* to $P\bar{P}$. The curves $g=$const. are *orthogonal trajectories* to the bridges as well as their images, that is, *circles having O for center.* The transformation is necessarily *reversible*, and the points of any bridge and their images determine an *involution.* The transformation is therefore an *inversion.*

§ 13. *Collineations.*

Assume now that a one-to-one transformation T admits *three invariant pencils* (A), (B), (C). *The indicatrices of the transformation form then a system of conics circumscribed to a fixed triangle ABC.*

Consider a line l of general position, and consider the family of indicatrices relative to the points on this line. They depend on but one parameter and consequently admit of an envelope (E).

Let E be the point in which the conic K relative to a point P touches the envelope. This point is uniquely determined, for by hypothesis *the indicatrix passes through three fixed points.* Let K' be the position of the indicatrix for an infinitely close point P'. The tactal τ of the line l for P is at the intersection of K and l, and likewise that of P' is at the intersection of K' and l. When the point P' approaches P, the point τ approaches the point E as a limit. The point E is therefore on l. Since all indicatrices relative to the line l can not touch the line l it follows therefore that the point E is fixed.

Consequently

Whenever a transformation admits three foci the family of indicatrices relative to the points on any straight line form a pencil of conics.

This theorem leads at once to the following:

Any trifocal transformation is a collineation.

Indeed the image of the line l has all its tangents going through the point E, it is therefore a line through E. We can complete then the preceding theorem by this statement. The four basic points of the pencil of conics are the three foci and the intersection of the line l and its image $\bar{l}$.

Among the indicatrices of the pencil, that relative to E is *tangent to the line l.* For let $\bar{E}$ be the image of E: the line $E\bar{E}$ is the bridge for E, consequently the line l must touch the indicatrix at l. (See Section 1.)

Conversely any collineation admits three foci. To show this I shall prove the following lemma:

The indicatrix of the image of a point P is the image of the indicatrix of the point P.

Indeed, let P and $\bar{P}$ be the two points. By hypothesis *any line l through P is transformed into a line $\bar{l}$ through $\bar{P}$.* Let l and $\bar{l}$ meet in a point τ. This point belongs to the indicatrix K of P. Let us seek the image $\bar{\tau}$ of τ. It is at the intersection of $\bar{l}$ and its image $\bar{\bar{l}}$, and consequently belongs to the indicatrix $\bar{K}$ of $\bar{P}$.

Assume now that the indicatrix K does not degenerate identically, that is that the collineation is *not central.* The two conics K and $\bar{K}$ meet besides in $\bar{P}$ in three other points A, B and C. The line PA goes into $\bar{P}A$, and $\bar{P}A$ into $\bar{\bar{P}}A$. *A is therefore an invariant point* and the same is evidently true of B and C. The three pencils (A), (B), (C) are therefore invariant. Consequently *any collineation is a trifocal transformation, and the indicatrices circumscribe a fixed triangle ABC.*

§ 14. *Perspective.*

In the proof of the last theorem we have expressly assumed that the indicatrix does not degenerate identically. Suppose now that the conic K degenerates at any point of the plane and let P and $\overline{P}$ be a point and its image. We know that the transformation is central, *i. e.*, $P\overline{P}$ goes through a fixed point P. Let β be the second branch of K. Any line l through P will intersect its image $\overline{l}$ in a point τ on β. It follows that if on the line l we take any other point Q the indicatrix of Q will also go through τ. And since τ is any point on β we conclude that

All indicatrices of a central collineation (*or perspective*) *have one branch in common* (*the axis of perspective*).

Any point on the axis β is a focus and the transformation possesses also an additional focus O without the axis. Let C be a point of the axis. The pencil (C) has for the two invariant lines CO and β.

The product of two perspectives is in general a non-central collineation. Indeed, let T_1 and T_2 be two perspectives with O_1 and O_2 as centers, and β_1 and β_2 as axes, respectively, and let P, P_1 and P_2 be a point, its image by T_1, and its image by T_1T_2, respectively. PP_1 goes through O_1, and P_1P_2 through O_2. According to a lemma proved in Section 9 the indicatrix of P by T_1T_2 is determined by P, $\overline{P}$ and the three intersections of the indicatrices of P and P_1 for T_1 and T_2, respectively. If then β_1 and β_2 meet in a point A, PP_1 and β_2 in F, and P_1P_2 and β_1 in G, the indicatrix of the resultant transformation will go through P, P_2, A, F and G, and consequently does not degenerate unless β_1 and β_2 coincide.

The perspectives of a plane do not form a group. All perspectives having a common axis form a group.

It is readily seen that the point A where the two axes of perspective meet is one of the foci of the resultant collineation. To find the other two foci we shall remark that T_1T_2 shifts O_1 and O_2 on the line O_1O_2. The line O_1O_2 is therefore invariant under T_1T_2 and contains the other two foci. The latter can therefore be determined as the intersections of the indicatrix for P and O_1O_2.

Conversely any collineation T can be regarded in an infinite number of ways as a product of two perspectives, T_1T_2. We can choose arbitrarily the two centers O_1 and O_2 on one of the sides of the invariant triangle, the two axes of perspective are then perfectly determined. They meet at the opposite vertex of the triangle. To actually perform the decomposition let P and $\overline{P}$ be

two corresponding points and let K be the indicatrix for P. O_1 and O_2 being taken at random on BC, let PO_1 meet K in a second point F, and $\overline{P}O_2$ meet K in G. Then FA and GA are the two axes of perspective. If now P_2 be the intersection of O_1P with $O_2\overline{P}$, P_1 is the image of P by T_1 while $\overline{P}$ is the image of P_1 by T_2. The two perspectives are therefore perfectly determined.

This theorem gives a first method of constructing a collineation given the invariant triangle and a pair of corresponding points.

§ 15. *Classification of Collineations.*

The indicatrix offers a natural method for the *classification* of *collineations.* Since the indicatrices of a collineation form a *bundle of conics* the following cases are possible:

A. *The three foci are distinct.* The indicatrices circumscribe the invariant triangle. We shall call such a collineation *trifocal.* From the standpoint of real transformations we must distinguish two cases: a) the three foci are all *real*, b) two of the foci are *conjugate imaginary.* In the first case all three invariant pencils are of the *hyperbolic* type. In the second case two of the pencils are *elliptic.*

B. *Two of the foci are coincident.* The indicatrices all pass through a fixed point A and touch a fixed line a at another fixed point B. The invariant pencil (B) is *hyperbolic* while (A) is *parabolic.* For any point of the line a the indicatrix degenerates into a and a line β passing through A; while on the other hand the indicatrix of any point on AB will have for the two branches AB and a line through B. In particular at B the indicatrix will consist of AB doubled. Therefore the point A belongs to the discriminant curve. We shall call this type a *bifocal* collineation.

C. All three foci are coincident in a point A. The collineation is *unifocal.* All indicatrices *osculate* at A and therefore have a common tangent a. The pencil (A) is parabolic and the point A again belongs to the discriminant curve.

D. One of the invariant pencils consists of invariant lines only. Suppose (C) to be this pencil; then C is a center and the opposite side c of the invariant triangle is a fixed line. The transformation is therefore a *perspective* having C for *center* and c for *axis.*

E. A particular case of the latter is when the center C is on c. In this case the transformation can be considered as a *degeneracy of the type C.* The curve c is a part of the discriminant curve. The collineation is an *elation.*

§ 16. *The Discriminant of a Collineation.*

The discriminant curve of a trifocal collineation is a parabola inscribed in the invariant triangle.

Consider indeed any line l and its image $\bar{l}$ and let E be their intersection. We have proved that the family of indicatrices relative to l is a pencil of conics having E for a *fourth fundamental point.* But among the conics of a pencil there are two parabolae: let D and D' be the points for which these parabolae are indicatrices. Then D and D' are the unique points in which l meets the discriminant curve Δ. The latter is therefore a conic.

When a point moves along any invariant side, say a, the second branch of its indicatrix turns about A; the correspondence thus generated between the range (a) and the pencil (A) is one-to-one. There exists therefore but one point L on a whose indicatrix degenerates into two parallel lines, *i. e.*, a and the line a' parallel to a and passing through A. This point L belongs to Δ and evidently a touches Δ at L. Since there is but one such point on any side of the invariant triangle, the conic Δ is *inscribed* in the triangle ABC.

On the other hand, the pencil of indicatrices relative to a line l generate a *parabolic* involution on l, since l goes through one of the basic points of the pencil and E is the double point. There exists therefore in a pencil but one conic K touching l, and the point of contact is in E. If we take for l the line at infinity, $\bar{\omega}$, the unique conic of the pencil relative to ω that touches $\bar{\omega}$ is evidently a parabola, and the point of contact belongs to the discriminant curve. The two points D and D' in which $\bar{\omega}$ meets Δ are coincident in E, and therefore D touches $\bar{\omega}$ at E. *The discriminant curve is thus a parabola.*

The asymptotic directions of the indicatrix at a point are the two tangents drawn from P to the discriminant conic Δ.

Indeed, let l be a line through P parallel to its image $\bar{l}$. The point E is then at infinity, and the conics K of the pencil touching l at E are parabolas. The pencil has therefore two coincident parabolas and consequently l is tangent to the discriminant curve Δ.

As a corollary we see that *it is the interior of the parabola Δ that makes up the elliptic region of the plane.* If then the three foci are all real (hyperbolic) the invariant triangle is entirely within the region H. If two of the foci are conjugate imaginary, the real focus, being elliptic, must be within the parabola.

As a second obvious corollary we derive that *the bridges relative to the vanishing line ω envelope the discriminant curve.* It follows from this that the

vanishing line itself *is a tangent to* Δ, for there exists on it one point $\vee$ which has for image the point at infinity on ω, and since the image of the bridge of a point touches the indicatrix at the image of the point, $\vee$ has for indicatrix a parabola and is therefore on Δ.

The same considerations hold with but slight modifications in the case of *bifocal and unifocal* collineation. If A and B are the two foci, and a and c the two invariant lines, the curve will be *inscribed in the angle* (a, c) touching c at A. In the case of a *unifocal* collineation the parabola Δ is osculating at A all indicatrices, it is therefore itself an indicatrix.

In a perspective the conic Δ *degenerates in a double line parallel to the axis of perspective and going through the center. This line coincides with the axis of perspective in the case of elation.*

§ 17. *The Conformal Points of a Collineation.*

Let l and l' be any two lines through a point P, and $\bar{l}$ and $\bar{l}'$ their images through $\bar{P}$. If the angles (l, l') and $(\bar{l}, \bar{l}')$ are equal for all positions of the lines, I shall call P a *conformal point.* I shall prove now that

There exists for any non-conformal collineation one and but one conformal point in the elliptic region. This point is the focus of the discriminating parabola.

Indeed, to determine the point it is sufficient to find the point E which has for indicatrix *the circle circumscribed to the invariant triangle*, and providing the collineation is not conformal, there exists but one such point. Now the asymptotic directions at E are *isotropic*, and since they are, according to the preceding section tangents to Δ, E is the point from which we can draw two isotropic tangents to Δ, that is, the *focus* of Δ.

In the same way it can be shown that in the *hyperbolic* region of the plane there also exists a point H at which angles are conserved. The indicatrix for this point must be an equilateral hyperbola and $H\bar{H}$ a diameter of the indicatrix. Now, since the asymptotic directions of all indicatrices are tangent to Δ, the locus of the point for which the indicatrix is a rectangular hyperbola is the *directrix* d *of the discriminating parabola.* The line d goes through the *orthocenter* of the invariant triangle and meets there its image $\bar{d}$, for all *equilateral hyperbola circumscribed to a triangle pass through its orthocenter. The point* H *may be any point on* d, *but once selected, the collineation is perfectly determined.* Indeed the indicatrix is then known by five points (H, orthocenter, and A, B, C) and consequently by taking for $\bar{H}$ the point diamet-

rically opposed to H on the indicatrix, the indicatrix is determined by the three invariant points and a pair of corresponding points.

The actual construction of the point $\bar{H}$ can be effected in the following manner: Let O be the orthocenter of the triangle. Then OA must be seen from the same angle from H and $\bar{H}$. If then we construct on OA *an arc of capacity* OHA the point $\bar{H}$ is on the arc, providing the arc has been so drawn that the angles $O\bar{H}A$ and OHA have opposite senses of rotation. If we now perform the same construction for OB, the point $\bar{H}$ will be determined as one of the intersections of the two circles, the other being in O.

If a collineation admits more than one conformal point of each type it is a conformal transformation. This is clear for the elliptic points, for were there more than one such point, the corresponding indicatrices being circles, two of the invariant points would be the cyclic points at infinity, and consequently all indicatrices would be circles. As to the hyperbolic conformal points the case can be immediately reduced to the preceding one by a reflection. *A non-central hyperbolic conformal collineation can not exist however, for all conics circumscribing a triangle can not be equilateral hyperbolae.*

§ 18. *Geometrical Determination of a Collineation.*

LEMMA. *If P, $\bar{P}$ and Q, $\bar{Q}$ are two pairs of corresponding points, and if the indicatrices of the collineation for P and Q meet in a fourth point S, then will PQ and $\bar{P}\bar{Q}$ meet in S.*

PROBLEM 1. *Determine a collineation by its invariant triangle and two pairs of corresponding points P, $\bar{P}$ on a and Q, $\bar{Q}$ on b.*

The second branch of the indicatrix for P passes through A, and that of Q through B, and according to the lemma both indicatrices contain the point S in which PQ and $\bar{P}\bar{Q}$ meet; the indicatrices are therefore perfectly determined. In order now to obtain the image of any point R in the plane, join R to P and let RP meet SA in R_a, also draw RQ meeting SB in R_b; then will $\bar{P}R_a$ and $\bar{Q}R_b$ meet in the image $\bar{R}$.

PROBLEM 2. *Determine a collineation by its invariant triangle and one pair of corresponding points R and $\bar{R}$ not belonging to the invariant triangle.*

We can immediately reduce this to Problem 1 by drawing RA, $\bar{R}A$, RB, $\bar{R}B$; for if the first two lines meet a in P and $\bar{P}$, and the second two meet b in Q and $\bar{Q}$, P and $\bar{P}$, Q and $\bar{Q}$ are evidently corresponding points.

The problem can also be treated directly, for the indicatrix K_R of R is determined by the five points R, $\bar{R}$, A, B and C. We can therefore determine projectively the intersection of any line RS with K. Let this intersection be τ; then the indicatrix K_S of S is perfectly determined by the five points τ, S, A, B, C, and the determination of $\bar{S}$ requires the finding of the intersection of $\bar{R}$ with K_S. The construction therefore necessitates two Pascal constructions.*

PROBLEM 3. *Determine a collineation by four pairs of corresponding points of general position:* $P, \bar{P};\ Q, \bar{Q};\ R, \bar{R};\ S, \bar{S}$.

If we denote by $PQ \cdot \bar{P}\bar{Q}$ the intersection of PQ and $\bar{P}\bar{Q}$, the conics K_P and K_Q are determined respectively by the five points

$$P;\ \bar{P};\ PQ \cdot \bar{P}\bar{Q};\ PR \cdot \bar{P}\bar{R};\ PS \cdot \bar{P}\bar{S} \text{ and } Q;\ \bar{Q};\ QP \cdot \bar{Q}\bar{P};\ QR \cdot \bar{Q}\bar{R};\ QS \cdot \bar{Q}\bar{S}.$$

By one Pascal construction we can therefore determine the intersection τ_p of K_P with the line MP, M being any point in the plane. A second Pascal construction would give us the point τ_q where MQ meets K_Q. The point $\bar{M}$ is at the intersection of $\bar{P}\tau_p$ and $\bar{Q}\tau_q$.

We have the following theorem as a consequence of the fact that *four pairs of points determine a collineation.*

Let $P, \bar{P};\ Q, \bar{Q};\ R, \bar{R};\ S, \bar{S}$ *be four pairs of points of general position. The four conics* K_P, K_Q, K_R *and* K_S *determined respectively by*

$$P, \bar{P}, PQ \cdot \bar{P}\bar{Q}, PR \cdot \bar{P}\bar{R}, PS \cdot \bar{P}\bar{S};\ Q, \bar{Q}, QP \cdot \bar{Q}\bar{P}, QR \cdot \bar{Q}\bar{R}, QS \cdot \bar{Q}\bar{S};$$

$$R, \bar{R}, RP \cdot \bar{R}\bar{P}, RQ \cdot \bar{R}\bar{Q}, RS \cdot \bar{R}\bar{S};\ S, \bar{S}, SP \cdot \bar{S}\bar{P},\ SQ \cdot \bar{S}\bar{Q}, SR \cdot \bar{S}\bar{R}$$

have three points in common.

PROBLEM 4. *A collineation being given by its invariant triangle and a pair of corresponding points,* P *and* $\bar{P}$, *determines its discriminating parabola* Δ *and the elliptic conformal point.*

Applying again our lemma let the parallel to a through A meet the indicatrix K_P in S, then if PS meets a in L, L is the point where a touches Δ. The points M and N are determined in a similar manner.

The *elliptic conformal point* E is located by the following simple construction. Let the parallel through A meet the circumscribing circle K_E in τ, and let τL meet the circle in a second point E. The latter is evidently the *conformal point.*

The latter construction can be applied to the problem: *Find the focus of a parabola inscribed in a triangle.*

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*I call Pascal construction the construction based on Pascal's hexagram theorem.

§ 19. *Displacements, Linear Transformations.*

A *conformal* collineation must be either of the *trifocal* type or a *central* collineation.

I. *Rotation.* If the indicatrices of a conformal collineation do not degenerate identically, they can not consist of equilateral hyperbolae only. They are therefore circles, and two of the invariant points are the cyclic points I and J. The transformation is a *rotation. All indicatrices have a common radical center O, the center of rotation.* If the transformation is *real* this latter is *real*, and the *bundle of circles* is a *hyperbolic* bundle. If the center be taken for origin and if α be the angle of rotation, the equation of the transformation is

$$\overline{x}=x\cos\alpha-y\sin\alpha,\quad \overline{y}=x\sin\alpha+y\cos\alpha. \tag{27}$$

The equation of the indicatrix is then

$$X^2+Y^2=k(xX+yY),\ \text{where}\ k=\sec\alpha. \tag{28}$$

II. Still assuming that the *conformity is direct* if the indicatrix degenerates identically, this can only happen in two ways. Either the two branches are *asymptotic* directions of the circle, that is *isotropic* lines and then the transformation is imaginary; or *one of the branches is the line at infinity.* Any line l has in the latter case a *parallel* image. Assume first that the *center of perspective* is not on the *line at infinity.* The transformation is then *homothetic.*

III. If, however, the perspective becomes an *elation*, that is its center is also on the line at infinity, the transformation is a *translation.*

IV. Finally, if the conformity of the collineation is of the *hyperbolic* type, the indicatrix must consist of two perpendicular straight lines, and its *fixed branch must bisect the bridge.* The bridges are therefore all parallel and the *center of perspective is at infinity.* We have then a *reflection.*

More generally consider a collineation leaving the line at infinity invariant. If the two invariant points on the line at infinity are real, all the indicatrices are hyperbolae with *fixed asymptotic directions.* If, on the other hand, the two foci at infinity are imaginary, the transformation is elliptic throughout the plane. The collineation in both cases are *linear transformations*, and, if the finite focus be taken for origin, admit the equations

$$\overline{x}=ax+by,\quad \overline{y}=cx+dy. \tag{29}$$

The discriminant is

$$\Delta=(a-d)^2+4bc, \tag{30}$$

so that the transformation is hyperbolic or elliptic according as Δ is positive or negative. If $\Delta=0$ the two foci at infinity become coincident. The indicatrices represent a bundle of parabolae passing through a fixed point, and admitting at the point a fixed diameter. Geometrically the three cases are distinguished by the fact that a hyperbolic linear transformation will leave two real pencils of parallel lines invariant, a parabolic but one real pencil while an *elliptic linear transformation will not move any real line parallel to itself.*

The last considerations are capable of an obvious generalization. Consider any *Magnus* transformation that admits *two foci at infinity.*

The indicatrices of such a transformation will have fixed asymptotic directions, and, conversely, any one-to-one transformation admitting a system of indicatrices with fixed asymptotic directions is a Magnus transformation with two foci at infinity.

If the third fundamental point be taken for origin and the directions determined by the foci as directions of coordinate axes the equation of such a Magnus transformation takes the form

$$\bar{x}=A/(x-a), \quad \bar{y}=B/(y-b). \tag{31}$$

A and B being arbitrary coefficients, and a, b the coordinates of the finite fundamental point.

§ 20. *Dualistic Developments.*

The results reached in this investigation admit of obvious *dualistic developments.*

Given a *line-to-line transformation* T, let l and $\bar{l}$ be two corresponding lines. Consider a line configuration C admitting l as a simple tangent, and let P be the point of contact. The image $\bar{C}$ of C will touch $\bar{l}$ at a unique point $\bar{P}$. I shall continue to call $P\bar{P}$ the *bridge.* The bridge is independent of the curve C chosen and depends only on the line l and the point P. When the point P describes l there is a *one-to-one correspondence* generated between the ranges l and $\bar{l}$, and the *envelope of the bridge is consequently a conic* L touching l and $\bar{l}$. I shall call *this conic the indicatrix of the line transformation* T *for the line* l.

The discussion of the divers cases brings out the following properties of the *line indicatrix* which I state without proof:

Let S be the intersection of l and $\bar{l}$, and let L touch l and $\bar{l}$ in A and B, respectively, and furthermore let A' and B' be the points of L diametrically opposed to A and B, then

1°. *If a curve C touches l in A its image $\overline{C}$ will touch $\overline{l}$ in S, and conversely: if C touches l in S then will $\overline{C}$ touch $\overline{l}$ in B.*

2°. *If a curve C admits l for asymptotic, its image $\overline{C}$ will touch $\overline{l}$ in A', and if C touches l in B', $\overline{l}$ is an asymptotic of $\overline{C}$.*

3°. *Let C and C' be any two curves and l a common tangent, touching C in P, C' in P'. If $\overline{C}$ and $\overline{C}'$ touch $\overline{l}$ in $\overline{P}$ and $\overline{P}'$, I shall say that distances are conserved along l and $\overline{l}$ if $\overline{PP}'=PP'$.* A line transformation that conserves distances throughout the plane I shall call a *collateral transformation*, directly collateral if $\overline{PP}'=PP'$ inversely if $\overline{PP}'=-PP'$. Then *the condition necessary and sufficient that a transformation be inversely collateral is that the indicatrix be a parabola symmetrically inscribed in the angle $(l, \overline{l})$. For a direct collateral transformation it is necessary and sufficient that the indicatrix degenerate into two points, the point S where l and $\overline{l}$ meet, and the point at infinity in the direction of a bisector of $(l, \overline{l})$.*

4°. If the indicatrix for a line l degenerates, three cases are possible: A. *l is an invariant line;* B. *l is an element of the Jacobian;* C. *Any curve touching l at S will have an image touching $\overline{l}$ at S.* The degenerate indicatrix consists of two points, which in Case A are both on l; in Case B one of the points is on l, the other on $\overline{l}$; finally, in Case C, one of the points is in S.

5°. *If the indicatrix degenerates identically and T is a proper transformation the point S will describe a curve, which may again be called the directrix of T, and T a directed transformation.* If besides T is one-to-one, the directrix is necessarily a *straight line* which I shall call *the axis* of T, and T an *axial transformation.*

6°. If a range of points goes into itself the bearer of the range, f, is a *focal line.* All the indicatrices of the plane will touch f. We can prove in the same way in the correlative case that *a one-to-one transformation will not admit more than three non-concurrent focals.* In the case of three focals we have a collineation which is a *self-dualistic* transformation. If, however, three focals are concurrent in a point O, any line through O is a focal and the transformation is a *perspective.* The latter being an axial transformation, any indicatrix will consist of the point O and a second point on the axis of perspective.

7°. If a collineation T be regarded as a line transformation all *indicatrices are inscribed in the invariant triangle. The discriminating parabola of T regarded as a point collineation is nothing but the indicatrix for the line at infinity of T regarded as a line configuration.*

This outline may suffice to bring out the most prominent features of the analogy between point and line transformations.

§ 21. *Extensions and Generalizations.*

We have expressly assumed in the preceding investigations that our transformation is *of multiplicity one-to-one.* I shall reserve for a subsequent paper the detailed study of *transformations of higher multiplicity.* Nevertheless I think this is the proper place to point out that most of the results reached apply without modifications to the general case.

Let

$$U(\bar{x}, \bar{y}, x, y)=0, \quad V(\bar{x}, \bar{y}, x, y)=0, \tag{32}$$

be the equations of the transformation in implicit form, and let $P(x, y)$ be any point in the plane. If, regarding x and y as constants, the solution of equations (32) give a *discrete number of values for* $\bar{x}$ *and* $\bar{y}$ *all distinct* we shall say that the point P is *regular* with respect to the transformation T. If the point is not a regular point it is either a *singular* point, that is, it has an *infinity of corresponding points,* or it is a *point of coincidence,* that is, at least two of its images are coincident.

All our investigations conserving the indicatrix apply evidently to any continuous point to point transformation at any regular point. If the point P has n distinct images $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_N$, *there will be n indicatrices passing through P, one for each of its images.* Once the couple of points (x, y) and $(\bar{x}, \bar{y})$ determined, the equation of the indicatrix becomes

$$\begin{vmatrix} \dfrac{\partial U}{\partial x}(X-x)+\dfrac{\partial U}{\partial y}(Y-y), & \dfrac{\partial V}{\partial x}(X-x)+\dfrac{\partial V}{\partial y}(Y-y), \\ \dfrac{\partial U}{\partial \bar{x}}(X-\bar{x})+\dfrac{\partial U}{\partial \bar{y}}(Y-\bar{y}), & \dfrac{\partial V}{\partial \bar{x}}(X-\bar{x})+\dfrac{\partial V}{\partial \bar{y}}(Y-\bar{y}), \end{vmatrix}=0. \tag{33}$$

Assume now that the point P *is a point of coincidence.* For example, that T is a two-to-one transformation and the two images of P are coincident in $\bar{P}$. It is clear then that at P the image of any curve C passing through P will have a double point. To a ray t through P will correspond two rays through $\bar{P}$, but to a ray $\bar{t}$, but one ray t will correspond. The indicatrix is therefore cut by any ray through P in one point only. The indicatrix is therefore a cubic having a double point in $\bar{P}$ and passing through P. On the other hand, this cubic is but a *limiting* position of a configuration consisting of two conics, it must therefore degenerate and it consists of *three straight lines* α, β,

γ, of which α and β meet in $\overline{P}$. The point P is evidently on the Jacobian of T. At any other point of the plane the behavior of the indicatrix is regular, and *any determination of T can be regarded as a one-to-one transformation.* All the considerations developed in the general theory apply therefore with this restriction.

In the most general case of a continuous p-to-q transformation the behavior of the indicatrix at a point will be analogous to that of the example considered. As long as we avoid the singularities described above, all the results of the general theory hold.

* * *

As to the generalization of the theory, it can be pursued in two directions. First, we may extend the conception of the indicatrix to any *surface of genus zero;* second, to *space line-to-line transformations.* I reserve these considerations for a subsequent communication.

BLOOMINGTON, IND., *February*, 1917.

Properties of a Certain Projectively Defined Two-Parameter Family of Curves on a General Surface.

By Pauline Sperry.

1. *Analytic Foundation of the Differential Geometry of Non-Ruled Surfaces.*

In his first memoir on the "Projective Differential Geometry of Curved Surfaces,"* Mr. Wilczynski has shown that the projective theory of non-ruled analytic surfaces may be based on the consideration of a system of completely integrable partial differential equations of the second order which may be reduced to the so-called canonical form

$$y_{uu}+2by_v+fy=0, \quad y_{vv}+2a'y_u+gy=0, \tag{1}$$

where the subscripts denote partial differentiation, and where the coefficients are analytic functions of u and v satisfying the integrability conditions

$$\left.\begin{aligned} &a'_{uu}+g_u+2ba'_v+4a'b_v=0, \quad b_{vv}+f_v+2a'b_u+4ba'_u=0,\\ &g_{uu}-f_{vv}-4fa'_u-2a'f_u+4gb_v+2bg_v=0. \end{aligned}\right\} \tag{2}$$

Such a system of differential equations possesses exactly four linearly independent analytic solutions

$$y^{(k)}=f^{(k)}(u, v) \qquad (k=1, 2, 3, 4). \tag{3}$$

If we now interpret $y^{(1)}, \ldots, y^{(4)}$, as the homogeneous coordinates of a point P_y, and let the independent variables range over all their values, P_y will generate a surface S_y, an integral surface of (1), which will be a ruled surface if, and only if, $a'=0$ or $b=0$ (a case which we shall exclude in this paper), and upon which the reference curves, $u=$constant and $v=$constant, are the

* There are five of these memoirs which appeared in the *Transactions of the American Mathematical Society* from 1907–1909. These will be referred to in the following pages as "First Memoir," etc.

asymptotic lines. Since the most general system of linearly independent solutions of (1) is of the form

$$\eta_i=\sum_{k=1}^{4} c_{ik}y^{(k)} \qquad (i=1, 2, 3, 4), \tag{4}$$

where $|c_{ik}|\neq 0$, the most general integrating surface of (1) is a projective transformation of any particular one.*

The canonical form is not uniquely determined. The most general transformation leaving it invariant will preserve the asymptotic curves as lines of reference and will be of the form

$$\bar{y}=C\sqrt{\alpha_u\beta_v}\,y, \quad \bar{u}=\alpha(u), \quad \bar{v}=\beta(v), \tag{5}$$

where α and β are arbitrary functions of u alone and of v alone respectively, and where C is an arbitrary constant.†

The functions

$$y=y, \quad y_u=z, \quad y_v=\rho, \quad y_{uv}=\sigma \tag{6}$$

are semicovariants. If the four independent solutions of (1) are substituted in z we get four functions $z^{(1)}, \ldots, z^{(4)}$, which may be taken as the homogeneous coordinates of a point P_z. So also for ρ and σ. The points P_y, P_z, P_ρ, P_σ are in general the vertices of a non-degenerate semicovariant tetrahedron T.‡ In just the same way every expression of the form

$$x=x_1y+x_2z+x_3\rho+x_4\sigma \tag{7}$$

determines a point P_x whose coordinates referred to T, by means of a suitable choice of the unit point, may be taken as (x_1, x_2, x_3, x_4).

2. *The Differential Equation of Certain Two-Parameter Families of Curves on a General Surface.*

The theory of two-parameter families of curves on a general surface has received but little attention except in so far as such a general theory may be implied by the theory of geodesics. We shall discuss in this paper a class of curves which will include the geodesics as a special case.

Let us associate with every point P_y of the surface one of the lines l_y which passes through that point, but does not lie in the tangent plane of the point. All these lines form a congruence L. Let us consider a curve on the surface which has the property that each of its osculating planes passes through the corresponding line of the congruence. All such curves will clearly

* First Memoir, p. 237. † First Memoir, pp. 90–95. ‡ Second Memoir, pp. 79–80.

form a two-parameter family, and it is easy to show that they will be the integral curves of an equation of the form

$$u''v'-u'v''+2(bu'^3-a'v'^3)+2(p_1u'^2v'+p_2u'v'^2)=0, \tag{8}$$

where $u'=du/dt$, $u''=d^2u/dt^2$, etc., and where p_1 and p_2 are functions of u and v which depend upon the choice of the congruence L.

The coordinates of P_y referred to the local tetrahedron of reference T are $(1, 0, 0, 0)$ so that the equation of any line l_y through P_y referred to T may be written in the form

$$A_2x_2+A_3x_3+A_4x_4=0, \quad B_2x_2+B_3x_3+B_4x_4=0, \tag{9}$$

where the coefficients are in general functions of u and v. If we denote by Π_1 and Π_2 the left-hand members of (9),

$$\lambda\Pi_1+\mu\Pi_2=0 \tag{10}$$

is the equation of any plane through l_y. If, in particular, this plane is the osculating plane of a surface curve through P_y, the coordinates of y' and y'' must also satisfy (10). By means of (6),

$$\left.\begin{aligned} y' &=dy/dt=u'z+v'\rho, \\ y''&=d^2y/dt^2=-(u'^2f+v'^2g)y+(u''-2a'v'^2)z+(v''-2bu'^2)\rho+2u'v'\sigma. \end{aligned}\right\} \tag{11}$$

The coordinates of y' and y'' referred to T are therefore

$$(0, u', v', 0), \quad (-u'^2f-v'^2g, \ u''-2a'v'^2, \ v''-2bu'^2, \ 2u'v').$$

The substitution of these coordinates in (10) gives upon the elimination of the ratio $\lambda:\mu$ equation (8), where

$$p_1=\frac{A_2B_4-A_4B_2}{A_3B_2-A_2B_3}, \quad p_2=\frac{A_3B_4-A_4B_3}{A_3B_2-A_2B_3}.$$

It is now evident that the line l_y must not lie in the tangent plane, as the denominator $A_3B_2-A_2B_3$ would vanish in that case and only in that case. If v be taken as the parameter of the curve, equation (8) will become

$$u''+2bu'^3+2p_1u'^2+2p_2u'-2a'=0. \tag{12}$$

It is also true that the one-parameter families of planes osculating the integral curves of an equation of the form (8) at the point P_y form a pencil. The equation of such a plane is

$$v'x_2-u'x_3+(p_1u'+p_2v')x_4=0. \tag{13}$$

If (u_1', v_1'), (u_2', v_2'), (u_3', v_3') are three pairs of values of u', v' corresponding to three distinct curves of this sort which pass through the same point, it is

evident that the corresponding equations (13) are not linearly independent for the determinant of the coefficients

$$\begin{vmatrix} v_1' & -u_1' & p_1u_1'+p_2v_1' \\ v_2' & -u_2' & p_1u_2'+p_2v_2' \\ v_3' & -u_3' & p_1u_3'+p_2v_3' \end{vmatrix}$$

vanishes identically. We have proved the following theorem:

If a two-parameter family of curves on a non-ruled surface has the property that the osculating planes of all of the curves of the family which pass through a given surface point P_y have in common a line through P_y, then the second order differential equation of the two-parameter family of curves has the form (8) *and conversely.*

These curves we have called *union curves* because of the characteristic property of united position of line and plane. It is evident that neither of the one-parameter families of asymptotic curves u=constant and v=constant can be union curves on a non-ruled surface since that would necessitate the condition $a'=0$, or $b=0$.

Among the congruences associated with the surface in the way described above are two of particular interest, the congruence of surface normals and the congruence of directrices of the second kind. One may define a geodesic as the curve whose osculating plane at a point contains the surface normal for that point. In case that for a particular one of the projectively equivalent integral surfaces of (1), the congruence L happens to be the congruence of surface normals, the corresponding union curves on that surface will be geodesics. Since perpendicularity is not a projective property, they would not in general be geodesics on the other integral surfaces of (1). The other congruence, the congruence of directrices of the second kind, is determined as follows: If we take five consecutive tangents to the curve v=constant, they determine in general a linear complex which approaches a limiting position as the tangents approach coincidence.* This complex is called the *osculating linear complex* for the asymptotic curve of the first kind. Similarly for the asymptotic curve of the second kind u=constant.† These complexes have as their intersection a congruence, one of whose directrices lies in the tangent plane of the point P_y, while the other, known as *directrix of the second kind,*

* E. J. Wilczynski, "Projective Differential Geometry," p. 162. In the following pages this book will be referred to as Proj. Diff. Geom.

† Second Memoir, pp. 90–95.

passes through the point itself, but does not lie in the tangent plane. The equations of the latter referred to the local tetrahedron T are

$$2bx_2+b_vx_4=0, \quad 2a'x_3+a'_ux_4=0. \tag{14}$$

If the congruence L is the directrix congruence of the second kind,

$$p_1=-\alpha, \quad p_2=\beta, \tag{15}$$

where

$$\alpha=a'_u/2a', \quad \beta=b_v/2b. \tag{16}$$

Equation (12) is a non-linear, non-homogeneous equation of the second order whose coefficients are functions of u and v. Since the complete solution involves two arbitrary constants, it represents a two-parameter family of curves, any one of which is uniquely determined when the surface point through which it must pass and the tangent at the point are given. The differential equation (12) can be integrated only in a few cases. The integration of an equation of this type in certain instances has been discussed by Darboux,* Liouville,† and Guldberg.‡ If L is the directrix congruence we can find a class of surfaces for which we can always integrate (12). Mr. Wilczynski has considered a class of surfaces for which the invariants $I=B_u/4BA^{\frac{1}{2}}$, and $J=A_v/4AB^{\frac{1}{2}}$, where $A=a'b^2$ and $B=a'^2b$, vanish identically, and he has shown that these surfaces are self-projective.§ For such surfaces we may assume without loss of generality that $a'=b=1$. Then equation (12) becomes $p'+2p^3-2=0$, where $p=u'$, and its first integral is

$$\log c\sqrt[6]{(p-1)^2/(p^2+p+1)}-(\sqrt[3]{3}/3)\ \text{arc tan}\ (2p+1)/\sqrt{3}=-2v.$$

3. *The Definition of Torsal Curves and Their Relation to Union Curves.*

There are two well-known fundamental properties of congruences. First, the lines of a congruence are the common tangents of two surfaces, or more precisely, they are the double tangents of a surface with two sheets, the focal surface. Second, if the two sheets of the focal surface do not coincide point for point, the lines of the congruence may be assembled into two one-parameter families of developables. We shall determine the curves on S_y such that the ruled surface composed of the lines of L, corresponding to the points of these curves shall be developables. These curves we shall call *torsal curves.* There

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* Darboux, *Leçons,* Vol. III, Chap. 5.

† R. Liouville, "Sur une classe d'équations différentielles," *Comptes Rendus,* Vol. CV (1887), p. 1062.

‡ A. Guldberg, "On Geodesic Lines on Special Surfaces," *Nyt Tydsskrift. Math.,* Vol. VI (1895). (See *Jahrbuch,* "Ueber die Fortschritte der Mathematik.")

§ E. J. Wilczynski, "On a Certain Class of Self-Projective Surfaces," *Transactions of the American Mathematical Society,* Vol. XLV (1913), pp. 421–443.

will be two of these curves passing through each non-special point of the surface, one from each family. We shall assume, therefore, that the two sheets of the focal surface are distinct, that is, that it will be impossible to find two functions $w_1(u, v)$ and $w_2(u, v)$ such that

$$w_1(u, v)\xi^{(k)} + w_2(u, v)\eta^{(k)} = 0 \qquad (k=1, 2, 3, 4),$$

where $\xi^{(k)}$ and $\eta^{(k)}$ are the coordinates of points on S_ξ and S_η, the two sheets of the focal surface.*

The equations of l_y, as given in (9), are satisfied by the coordinates $(0, -p_2, p_1, 1)$ so that the point P_t, where

$$t = -p_2 z + p_1 \rho + \sigma \tag{17}$$

is a point of l_y. If we allow u and v to take on the infinitesimal increments du and dv, the point P_y moves to the point $P_{y+y_u du+y_v d_v}$ and the point P_t to $P_{t+t_u du+t_v d_v}$ where

$$t_u = Py + Qz + R\rho + S\sigma, \quad t_v = P'y + Q'z + R'\rho + S'\sigma, \tag{18}$$

and

$$\left.\begin{aligned} P &= fp_2 + 2bg - f_v, & P' &= -gp_1 + 2a'f - g_u, \\ Q &= 4a'b - p_{2u}, & Q' &= -2a'p_1 - g - 2a'_u - p_{2v}, \\ R &= 2bp_2 - f - 2b_v + p_{1u}, & R' &= \;\; 4a'b + p_{1v}, \\ S &= p_1, & S' &= -p_2. \end{aligned}\right\} \tag{19}$$

An arbitrary point P_x on the line $l_{y+y_u du+y_v d_v}$ will be represented by

$$\lambda(y + y_u du + y_v dv) + \mu(t + t_u du + t_v dv),$$

and its coordinates referred to T will be

$$\left.\begin{aligned} x_1 &= \lambda + \mu(Pdu + P'dv), & x_2 &= \lambda du + \mu(-p_2 + Qdu + Q'dv), \\ x_3 &= \lambda dv + \mu(p_1 + Rdu + R'dv), & x_4 &= \mu(1 + Sdu + S'dv). \end{aligned}\right\} \tag{20}$$

In order that P_x may also be a point of l_y, its coordinates must satisfy (9), that is,

$$\left.\begin{aligned} \lambda[A_2 du + A_3 dv] + \mu[(A_2 Q + A_3 R + A_4 S)du + (A_2 Q' + A_3 R' + A_4 S')dv] &= 0, \\ \lambda[B_2 du + B_3 dv] + \mu[(B_2 Q + B_3 R + B_4 S)du + (B_2 Q' + B_3 R' + B_4 S')dv] &= 0, \end{aligned}\right\} \tag{21}$$

whence

$$\begin{vmatrix} A_2 du + A_3 dv & (A_2 Q + A_3 R + A_4 S)du + (A_2 Q' + A_3 R' + A_4 S')dv \\ B_2 du + B_3 dv & (B_2 Q + B_3 R + B_4 S)du + (B_2 Q' + B_3 R' + B_4 S')dv \end{vmatrix} = 0. \tag{22}$$

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* E. J. Wilczynski, "Sur la theorie general des congruences," Bruxelles, 1911.

If we let

$$L=2bp_2-f-p_1^2-2b_v+p_{1u}, \quad 2M=p_{1v}+p_{2u}, \quad N=2a'p_1+g+p_2^2+2a'_u+p_{2v}, \tag{23}$$

then (22) becomes

$$Ldu^2+2Mdudv+Ndv^2=0; \tag{24}$$

this is the quadratic equation of the tangents to the torsal curves. If these curves form a conjugate system, there exists a function θ, such that $p_1=-\theta_u$ and $p_2=\theta_v$. When L is the directrix congruence of the second kind, the torsal curves are the directrix curves.* If L is the congruence of normals, they are the lines of curvature.

The question naturally arises whether some or all of the union curves might be plane. We shall show that this can happen if, and only if, they are at the same time torsal curves.

Every curve in three-dimensional space is characterized by a linear differential equation of the fourth order of the form,†

$$q_0y^{(IV)}+4q_1y'''+6q_2y''+4q_3y'+q_4y=0, \tag{25}$$

where $y^{(i)}=d^iy/dt^i$ $(i=1, 2, 3, 4)$, and $q_0, \ldots, q_4$ are in general functions of t. We may regard the solutions of (25) as the homogeneous coordinates of a point in space. As t varies this point describes a curve. Since

$$\eta_i=\sum_{k=1}^{4} c_{ik}y_k \qquad (i=1, 2, 3, 4)$$

is the most general solution of (25), we may say the differential equation defines a set of projectively equivalent curves in three-dimensional space. These curves will be plane if, and only if, $q_0=0$. We shall indicate the method for computing the fourth order differential equation which characterizes the union curves and actually calculate the value of q_0. From equations (1) and (6), and those obtained from them by partial differentiation, and from the value of u'' given by (12), we find the following formulae in which v is chosen as independent variable:

$$\left.\begin{aligned} y' &=u'z+\rho, & y'' &=a_1y+a_2z+a_3\rho+a_4\sigma, \\ y''' &=b_1y+b_2z+b_3\rho+b_4\sigma, & y^{(IV)} &=c_1y+c_2z+c_3\rho+c_4\sigma, \end{aligned}\right\} \tag{26}$$

* Second Memoir, p. 116.

† Proj. Diff. Geom., Chap. 2.

where

$$\left.\begin{aligned}
&a_1=-fu'^2-g, \quad a_2=-2bu'^3-2p_1u'^2-2p_2u', \quad a_3=-2bu'^2, \quad a_4=2u',\\
&b_1=\ \ 6bfu'^4+(6fp_1-f_u)u'^3+3(2bg+2fp_2-f_v)u'^2-3g_uu'-g_v,\\
&b_2=-fu'^3+12a'bu'^2,\\
&b_3=\ \ 12b^2u'^4+(12bp_1-2b_u)u'^3+3(4bp_2-f-2b_v)u'^2-g,\\
&b_4=-8bu'^3-6p_1u'^2-6p_2u'+4a',\\
&c_1=[(2bg-f_v)b_4-fb_2+b_{1u}]u'-gb_3+(2a'f-g_u)b_4+b_{1v},\\
&c_2=(b_1+4a'bb_4+b_{2u})u'-2a'b_3-(g+2a'_u)b_4+b_{2v},\\
&c_3=[-2bb_2+b_{3u}-(f+2b_v)b_4]u'+b_1+4a'bb_4+b_{3v},\\
&c_4=(b_3+b_{4u})u'+b_2+b_{4v},
\end{aligned}\right\} \qquad (27)$$

where $b_{1u}, \ldots, b_{4u}, b_{1v}, \ldots, b_{4v}$ are found from the fifth to eighth equations of (27) by partial differentiation, and u'' is given by (12). The desired equation will be obtained by eliminating z, ρ, and σ between equations (26) which gives

$$\begin{vmatrix} y' & u' & 1 & 0 \\ y''-a_1y & a_2 & a_3 & a_4 \\ y'''-b_1y & b_2 & b_3 & b_4 \\ y^{(\mathrm{IV})}-c_1y & c_2 & c_3 & c_4 \end{vmatrix}=0. \qquad (28)$$

Thus the coefficient of $y^{(\mathrm{IV})}$ is

$$q_0=-\begin{vmatrix} u' & 1 & 0 \\ a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{vmatrix},$$

or, substituting from (27),

$$q_0=4u'^2(Lu'^2+2Mu'+N). \qquad (29)$$

Since u' can not vanish, $q_0=0$ is equivalent to (24). Hence we have proved the following theorem:

A necessary and sufficient condition that the union curves be plane is that they shall also be torsal curves.

This necessitates two relations between p_1, p_2, a', b and their derivatives which are obtained by the substitution in (12) of each of the pairs of values of u' and u'' found from (23) and the equation obtained from it by differentiation. If we let

$$R^2=M^2-LN, \qquad (30)$$

these relations are

$$\left.\begin{aligned} &2a'L^3+8bM^3+2M^2(L_u-2p_1L)+L^2(2p_1N+2p_2M+M_v+\tfrac{1}{2}N_u) \\ &\qquad\qquad -LM(2M_u+L_v)-LN(6bM-\tfrac{1}{2}L_u)=0, \\ &R(4bM^2+2bLMN-4p_1LM+2p_2L^2-LM_u+2ML_u-LL_v) \\ &\qquad\qquad +L^2R_v-LMR_u=0. \end{aligned}\right\} \quad (31)$$

The second of these is satisfied identically when $R=0$, that is, when the two families of torsal curves coincide. For this case (31) reduces to the single condition

$$2a'L^2+2bM^3-2p_1LM^2+2p_2L^2M-LMM_u+L_uM^2+L^2M_v-LL_vM=0, \quad (32)$$

for particular values of p_1 and p_2, one must always show that these conditions are not inconsistent with the integrability conditions (2). This caution applies also to all similar situations which occur in the following pages.

Since by definition the osculating plane of a union curve at the point P_y contains the line l_y, and since the osculating plane of a plane curve is the same for all points of the curve, it is obvious geometrically that plane union curves are torsal curves, and that the developables are the planes of the curves themselves. If every curve of the two-parameter family of union curves is plane, it is evident that the torsal curves must be indeterminate, since there are only a single infinity of them. In that case we must have

$$L=M=N=0, \quad (33)$$

and conditions (31) would be satisfied identically.

In case all the union curves are plane, the lines of L for a restricted region R of the surface have a point in common as a simple geometric argument will show. Consider that part of a union curve C lying in R. It follows from the theory of differential equations that for R sufficiently small, there exist union curves joining two arbitrary points of C to a third point of R not on C. We obtain in this way an infinite number of triples of points for which the corresponding lines of L intersect in pairs. Since these lines are not all in the same plane, they must all pass through the same point, the edge of regression of all of the developables of the congruence.

The theorem proved above is of especial interest as it includes as a particular instance the well-known theorem that a geodesic is plane when, and only when, it is a line of curvature.

4. *The Principle of Duality Applied to Union Curves.*

The integral surface S_y of (1) may be regarded as the envelope of its tangent planes. It will then be represented by the partial differential equations.*

$$Y_{uu}-2bY_v+(2b_v+f)Y=0, \quad Y_{vv}-2a'Y_u+(2a'_u+g)Y=0, \tag{34}$$

where the solutions Y_i ($i=1, 2, 3, 4$) are proportional to the homogeneous coordinates of the plane p_y tangent to the surface S_y at the point P_y. If we let

$$\bar{a}'=-a', \quad \bar{b}=-b, \quad \bar{f}=2b_v+f, \quad \bar{g}=2a'_u+g, \tag{35}$$

we may replace (34) by

$$Y_{uu}+2\bar{b}Y_v+\bar{f}Y=0, \quad Y_{vv}+2\bar{a}'Y_u+\bar{g}Y=0, \tag{36}$$

which is called the system adjoined to (1). It is evident from (35) that each is the adjoint of the other. If the solutions Y_i ($i=1, 2, 3, 4$) are regarded as the coordinates of a plane, systems (1) and (36) have the same integral surfaces. But if they are regarded as the coordinates of a point, every integral surface S_Y of (36) would be a dualistic transform of every integral surface S_y of (1). Since $a' \neq 0$ and $b \neq 0$, S_y is not identically self-dual, that is, there exists no dualistic transformation carrying the point P_y over into the plane p_y tangent at that point. If Y_i be interpreted as the coordinates of a point, then the equation of the union curves on S_Y is

$$u''-2bu'^3+2\bar{p}_1u'^2+2\bar{p}_2u'+2a'=0. \tag{37}$$

In order to pass to the dualistic interpretation, let us regard S_y as the locus of its points, and S_Y as the envelope of its tangent planes. To the congruence L of lines l_y passing through the points of S_y, will correspond a congruence L_Y of lines l_Y in the tangent planes of S_Y. To a curve as point locus on S_y will correspond a developable circumscribed about the surface S_Y. Just as the union curves on S_y have the property that the osculating plane at P_y determined by three consecutive points of the curve contains the line l_y, so the point, which for symmetry we shall call the *osculating point for* p_y, determined by three consecutive planes of the developable, lies on the line l_Y. As the line l_y is the intersection of the osculating planes of all of the union curves passing through P_y, so also l_Y is the locus of all the osculating points of the developables containing p_y. To a plane union curve would correspond a cone with the osculating point as vertex. The theorems developed in the preceding pages are capable of dualistic interpretation, and could be developed independently by analysis dualistic to that employed above. The values of $\bar{p}_1$ and p_2

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* First Memoir, p. 257.

in (37) evidently depend upon the choice of L_Y. As instance of projectively related congruences of this character one may cite the directrix congruences of the first and second kind.

5. *The Ruled Surface of the Congruence L along a Union Curve.*

We shall determine the differential equations which characterize the ruled surface generated by the line l_y as P_y moves along one of the union curves. The coordinates of a point t of the line l_y given in (17), and of y, must satisfy differential equations of the form *

$$n_{11}y''+p_{11}y'+p_{12}t'+q_{11}y+q_{12}t=0, \quad n_{22}t''+p_{21}y'+p_{22}t'+q_{21}y+q_{22}t=0, \tag{38}$$

where $y'=dy/dv$, $y''=d^2y/dv^2$ and so on. By means of (1), (12), (17), (18) and (19) we find

$$\left.\begin{aligned} y &=y, \quad y'=u'z+\rho, \quad y''=-(u'^2f+g)y+(u''-2a')z-2bu'\rho-2u'\sigma, \\ t &=-p_2z+p_1\rho+\sigma, \\ t' &=(u'P+P')y+(u'Q+Q')z+(u'R+R')\rho+(u'S+S')\sigma, \\ t'' &=T_1y+T_2z+T_3\rho+T_4\sigma, \end{aligned}\right\} \tag{39}$$

where $T_1, \ldots, T_4$ are somewhat lengthy expressions in a', b, their partial derivatives and the powers of u' up to the third. By eliminating z, ρ and σ by means of the second, third, fourth and fifth of equations (39), and then from the second, fourth, fifth and sixth, there result two equations of type (38) where $n_{11}=n_{22}=-(Lu'^2+2Mu'+N)$. Since the ruled surface is a developable if, and only if, $n_{11}=n_{22}=0$, the results of § 3 are again emphasized.

The four pairs of independent solutions of (38), y_i, t_i, $(i=1,2,3,4)$ may be taken as the homogeneous coordinates of two points P_y and P_t. As the line l_y generates the ruled surface, these points describe curves C_y and C_t. The curve C_y is asymptotic for that surface provided that $p_{12}=0$.† The calculation of the coefficients gives

$$p_{12}=4u'(p_1u'+p_2). \tag{40}$$

Since $u'\neq 0$, $p_{12}=0$ when, and only when, $u'=-p_2/p_1$. This value of u' must satisfy (12) which imposes the following restriction

$$p_1p_2(p_{2u}+p_{1v})-p_2^2(p_{1u}+2bp_2)-p_1^2(p_{2v}+2a'p_1)=0. \tag{41}$$

That this condition may be satisfied is apparent. Let L be the congruence of directrices of the second kind, and let a' and b be functions of v alone, and of

* Proj. Diff. Geom., p. 126. † Proj. Diff. Geom., p. 142.

u alone, respectively. Then $p_1 = p_2 = 0$. Whenever p_1 and p_2 are such that (41) is satisfied, there exists a one-parameter family of union curves having the property that they are at the same time asymptotic curves of the ruled surface generated by the lines of L along those curves.

6. *Some Remarks on a Problem in the Calculus of Variations.*

One of the most interesting properties of geodesics is that they appear in the problem of determining the lines of shortest length on a surface. Because so much of the theory of geodesics has turned out to be capable of generalization to the theory of union curves, the question naturally arises whether there exists a function $F(u, v, u')$ such that the integral $\int F(u, v, u')\,dv$ assumes a minimum value along a union curve, and such that the integral is invariant under the transformation $\bar{u} = \alpha(u)$, $\bar{v} = \beta(v)$. It is possible to find an infinity of functions satisfying the first of these conditions. The latter condition is essential in order to obtain an integral which may have an intrinsic projective significance. Investigation of this question has yielded thus far only negative results.